THE ELEMENTS OF THE DIFFERENTIAL AND INTEGRAL CALCULUS

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THE ELEMENTS

OF THE

DIFFERENTIAL AND INTEGRAL

CALCULUS

WITH NUMEROUS EXAMPLES

BY

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PREFACE

This book was written to meet the needs of my own classes; yet it is hoped that not only teachers of mathematics in technical colleges, but those in classical colleges and universities as well, will find it suitable for a first course in the Differential and Integral Calculus.

In many technical colleges, among them the one with which I am connected, the study of Calculus is begun in the first year of the course. As such an arrangement involves beginning a difficult branch of mathematics with somewhat immature students, the first few chapters in both the Differential and Integral parts are discussed in more detail than is usual in text-books.

Throughout the book I have confined myself strictly to those subjects which I know from my own experience are most needed by my own students. It seemed wise to me to omit all subjects only remotely connected with those of engineering, and introduce a few elementary chapters in Mechanics. Thus I was able, without encumbering the book, to afford a short introduction to Mechanics and Differential Equations as well as to view the principles of Attraction, Centers of Gravity, and, to a certain extent, Moments of Inertia, from the mechanical rather than the purely mathematical side. If the teacher feels that he should treat any subject omitted here, he can readily do so by lecture.

The part of the book which differs most widely from other books is that dealing with the Integral Calculus. It has been my experience that a working knowledge of the principles of the Integral Calculus can be gained only by a careful con-

sideration of the details of the subject. I have, therefore, even at the risk of being considered prolix, entered into a full explanation of each step in the formation of each summation and integral.

It was my good fortune to begin the study of Mechanics and the Differential and Integral Calculus with MacGregor's *Kinematics and Dynamics* and Byerly's works in Calculus, under the respective authors. To these books and men I am indebted for help probably more than I am aware of being. I have not consciously followed any of their methods of treatment, however, although I frequently consulted the work in *Kinematics and Dynamics* during the preparation of the first two chapters in Mechanics. To Appel's *Course d'Analyse* I am indebted for the proof given in Art. 186 as well as for other valuable hints. To B. O. Peirce's *Table of Integrals*, Glanville Taylor's *Calculus*, Gibson's *Calculus*, Tait and Steele's *Dynamics of a Particle*, and Ziwet's *Mechanics* I also made frequent reference. A few problems I have taken from Harvard examination papers.

To a colleague, Professor N. C. Riggs, I am under obligations for an extremely careful revision of both the manuscript and proofs, for a number of problems with answers, and for verification of a large number of my answers. To another colleague, Professor C. W. Leigh, I am indebted for verification of a number of answers. To a third-year student of the Armour Institute of Technology, Mr. Garfield Lennartz, I am indebted for whatever of mechanical excellence the figures possess.

D. F. CAMPBELL.

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CONTENTS

CHAPTER X

CHAPTER XI

CHAPTER XII

CHAPTER XIII

CHAPTER XIV

CHAPTER XV

CHAPTER XVI

CHAPTER XVII

CHAPTER XVIII

CHAPTER XIX

CHAPTER XX

CHAPTER XXI

CHAPTER XXII

CHAPTER XXIII

CHAPTER XXIV

CHAPTER XXV

CHAPTER XXVI

CHAPTER XXVII

CHAPTER XXVIII

CHAPTER XXIX

CHAPTER XXX

CHAPTER XXXI

CHAPTER XXXII

THE ELEMENTS OF THE
DIFFERENTIAL AND INTEGRAL CALCULUS

DIFFERENTIAL AND INTEGRAL CALCULUS

INTRODUCTION

1. **Constants and variables.** A number that retains the same value throughout any given problem is called a **constant.**

A number that changes from one value to another in the same problem is called a **variable.**

For example, in the equation of the circle $x^2 + y^2 = a^2$, a is a constant and x and y are variables.

2. **Functions.** A number that so depends on a second number for its value that when the second is made to assume different values it, in general, assumes different values, is said to be a function of the second number.

For example, $a^2 - x^2$ is a function of x because, as x is made to assume different values, $a^2 - x^2$, in general, assumes different values.

It is said *in general* because sometimes for different values of the variables the function has the same value.

For example, in $a^2 - x^2$, for a value of x with the minus sign affixed and the same value of x with the plus sign affixed the function has the same value.

3. From the definition of a function it is seen that any combination of letters used to denote a number is a function of these letters. It may be called a function when the thought is of the combination of letters, and a number when the thought is of the value which this combination represents.

4. By a combination of letters is meant an expression which actually contains these letters and cannot be reduced to an equivalent expression which does not contain them.

Thus, $x^2 - y^2$ is a combination of x and y, since it contains x and y and cannot be reduced to an equivalent expression that does not contain them. The expression $(2+y)(2-y)+y\left(y+\frac{x}{y}\right)$ is not a combination of x and y because on simplification it reduces to $4+x$, which does not contain y.

5. A combination of letters is not a function of a letter not actually contained in the combination. This is obvious, because the expression evidently does not assume different values as the letter assumes different values.

Thus, $(2+y)(2-y)+y\left(y+\frac{x}{y}\right)$ is not a function of y.

6. **Independent and dependent variables.** In an equation containing two variables, if one of the variables is given a value chosen at pleasure, the other cannot be given a value chosen at pleasure. It assumes the value got by solving the equation. The one to which a value was given at pleasure is called the **independent** variable. The one that assumes the value got by solving the equation is called the **dependent** variable.

For example, in the equation $x^2 + y^2 = a^2$, suppose that x is given the value $\frac{a}{2}$. Then y must be such that $\frac{a^2}{4} + y^2 = a^2$, or $y = \pm\frac{a}{2}\sqrt{3}$. Here x is the independent variable and y the dependent variable.

It will be noticed that independent variable is merely another name for variable, and dependent variable another name for function.

7. **Classification of functions.** It is convenient for our purpose to make a rough classification of functions as follows:

Algebraic functions. Those in which the only operations performed upon the variable are a definite number of opera-

tions of addition, subtraction, multiplication, division, extraction of roots, or raising to powers.

Logarithmic functions. Those involving the logarithm of the variable or of an algebraic function of the variable.

Exponential or anti-logarithmic functions. Those involving an exponential function of the variable or of an algebraic function of the variable.

Trigonometric functions. Those involving a trigonometric function of the variable or of an algebraic function of the variable.

Anti-trigonometric functions. Those involving an anti-trigonometric function of the variable or of an algebraic function of the variable.

We may have combinations of these functions, as, for example, $\log_{10} \sin (\tan^{-1} x^2)$; but there would be no advantage in classifying such functions, and we shall not do so.

8. Algebraic functions. An algebraic function in any number of unknowns may be **integral** or **fractional, rational** or **irrational.**

An **integral algebraic function** is one in which, after all the negative exponents have been made positive, there are no variables in the denominator.

EXAMPLE. The expression $\sqrt[3]{x} + 2\,xy + 3\,y^2 + \sqrt{z}$ is an integral algebraic function.

A **fractional algebraic function** is one in which, after all the negative exponents have been made positive, there are variables in the denominator.

EXAMPLE. The expression $\sqrt[3]{x} + 2\,xy + \frac{3}{y^2} + \sqrt{z}$ is a fractional algebraic function.

A **rational algebraic function** is one in which the variables are all raised to powers whose exponents are integers, positive or negative.

Examples. The expressions x^2+y and $x^2+y^{-2}-4$ are rational algebraic functions.

An **irrational algebraic function** is one in which at least one power whose exponent is a fraction appears somewhere over the unknowns.

Examples. The expressions $x^2+y^{\frac{1}{2}}$ and $\sqrt{x^2+y^2}+xz$ are irrational algebraic functions.

9. It must be noticed that the character of the function is determined by the manner in which the variables, not the constants, are involved in the function.

Thus, $ax^2+\frac{1}{\sqrt{b}}xy^{\frac{1}{4}}+\frac{1}{c}y^2$ is integral if b and c are constant; fractional if b or c is variable; rational if b and y are constant; irrational if b or y is variable.

10. **Variables involved explicitly; implicitly.** In an equation in two variables, either variable is said to be involved **explicitly** when, for a given value of the other, its value can be found without solving an equation. It is said to be involved **implicitly** when, for a given value of the other, its value can be found only by solving an equation.

For example, in the equation $y=x^2+4x+1$, let x be given some value. Then the value of y corresponding to that value of x can be found without solving an equation. Then y is involved explicitly in the equation. If y is given some value, the corresponding value of x can be found only by solving the equation $x^2+4x=y-1$. Then x is involved implicitly in the equation.

11. It is generally, though not always, of advantage to arrange the variables so that the dependent variable is involved explicitly in the equation. Thus, in the equation $y=\sin x$, if y is regarded as the dependent variable, the equation would be written in the above form, while if x is regarded as the depend-

ent variable, the equation would most naturally be written as $x = \sin^{-1} y$. Or again, in $y = x^2$, if y is regarded as the dependent variable, the equation would be written in the above form, while if x is regarded as the dependent variable, the equation would most naturally be written as $x = \pm\sqrt{y}$.

12. **Functional symbols.** A function may be denoted by any convenient symbol, as f, F, ϕ, U, $\cdots$, followed by the variable or variables inclosed in brackets. If there are more variables than one in the function, the variables in the symbol, in general, are separated by commas. It is said *in general* because in a few rare cases they enter in a certain combination only. In these cases they may or may not be separated by commas.

For example, x^2+4 may be denoted by $f(x)$; $x^2+3xy+2y^2+6$ by $f(x, y)$, and $(xy)^2+4(xy)+1$, in which x and y appear in the combination xy, by $f(xy)$ or $f(x, y)$.

During the investigation of any particular problem the same functional symbol must be used to denote the same operation or series of operations.

For example, if $f(x) \equiv x^2+4$, then $f(y) \equiv y^2+4$, and $f(xy) \equiv (xy)^2+4$.

13. **Absolute value of real function.** By the absolute value of a real function is meant the value of the function disregarding the minus sign if the function is negative.

For example, the absolute value of $\frac{1}{x}$ when $x = -2$ is $\frac{1}{2}$.

Absolute value is denoted by two parallel lines. Thus, $\left|-\frac{1}{2}\right| = \frac{1}{2}$.

14. **Single-valued, double-valued, multiple-valued functions.** A function of one variable is said to be **single-valued** when for a value of the variable the function has just one value.

Thus, in the function given by the equation $y = \log_{10} x$, for a value of x, y or $\log_{10} x$ has just one value. Then y or $\log_{10} x$ is a single-valued function of x.

In the curve whose equation in Cartesian coördinates gives a single-valued function of x, any line parallel to the y-axis will cut the curve in just one point or not at all.

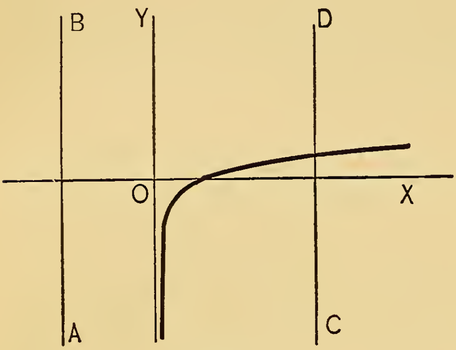


FIG. 1.

Thus, in the curve $y = \log_{10} x$ (see Fig. 1), any line, as AB or CD, parallel to the y-axis, cuts the curve in one point or not at all.

A function of one variable is said to be **double-valued** when, for a value of the variable, the function has two values.

Thus, in the equation $x^2 + y^2 = 4$, or $y = \pm\sqrt{4 - x^2}$, for a value of x, y or $\pm\sqrt{4 - x^2}$ has two values. Then y or $\pm\sqrt{4 - x^2}$ is a double-valued function of x.

In the curve whose equation in Cartesian coördinates gives a double-valued function of x, any line parallel to the y-axis will cut the curve in two points, distinct or coincident, or not at all.

Thus, in the curve $y = \pm\sqrt{4 - x^2}$ (see Fig. 2), any line, as AB or CD, parallel to the y-axis, cuts the curve in two points, distinct or coincident, or not at all.

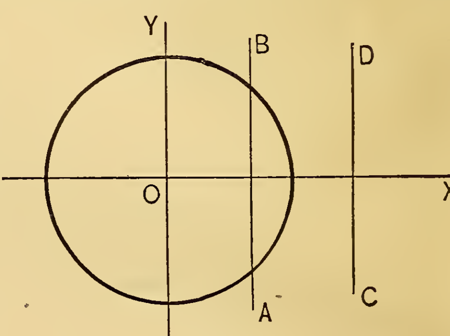


FIG. 2.

A function of one variable which has more than two values for a value of the variable is called a **multiple-valued function** of the variable.

Thus, in the equation $y = \tan^{-1} x$, y or $\tan^{-1} x$ has more than two values for a value of x. Then y or $\tan^{-1} x$ is a multiple-valued function of x.

In the curve whose equation in Cartesian coördinates gives

a multiple-valued function of x, any line parallel to the y-axis will usually cut the curve in more than two points.

Thus, in the curve $y = \tan^{-1}x$ (see Fig. 3), any line, as AB or CD, parallel to the y-axis, cuts the curve in an unlimited number of points.

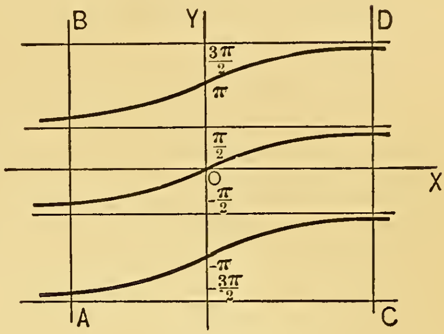


Fig. 3.

15. In any particular problem it is desirable to have the function $f(x)$ single-valued. If it is not single-valued, it can be made so by considering it as made up of a number of functions, each of which is single-valued. The curve whose equation gives the function will then consist of a number of branches, each of which will correspond to one of the component functions of $f(x)$.

Thus, in the equation $y = \pm\sqrt{4 - x^2}$, the function $\pm\sqrt{4 - x^2}$ may be considered as made up of the two functions $+\sqrt{4 - x^2}$ and $-\sqrt{4 - x^2}$, each of which is single-valued. In the curve $y = \pm\sqrt{4 - x^2}$ (see Fig. 2), the part above the x-axis corresponds to $y = +\sqrt{4 - x^2}$, and the part below to $y = -\sqrt{4 - x^2}$.

EXERCISES

1. For what values of n does $x^{n(n-1)}$ cease to be a function of x?

2. In the equation $y\left(1 + \frac{z + x}{z - x}\right) = \left(z + x + \frac{1 + x^2}{z - x}\right)$, show that y is a function of z but not of x.

3. Show that $\sin x \tan \frac{1}{2} x + \cos x$ is not a function of x.

4. In the equation $xy - 2x + y = n$, determine the value of n for which y ceases to be a function of x.

In each of the exercises 5 to 9 inclusive, express y explicitly in terms of x and a.

5. $\log_{10} x + \log_{10} y - 2\log_{10} a = 0$. *Ans.* $y = \dfrac{a^2}{x}$.

6. $3\log_{10} x - 2\log_{10} y + 4\log_{10} a = 0$. *Ans.* $y = \pm\sqrt{x^3a^4}$.

7. $\sin^{-1} x + \sin^{-1} y = a$. *Ans.* $y = \sqrt{1-x^2}\sin a - x\cos a$.

8. $\tan^{-1} x + \tan^{-1} y = a$. *Ans.* $y = \dfrac{\tan a - x}{1 + x\tan a}$.

9. $\cos^{-1} x + \tan^{-1} y = a$. *Ans.* $y = \dfrac{x\tan a - \sqrt{1-x^2}}{x + \sqrt{1-x^2}\tan a}$.

10. If $f(x) \equiv x^3 - 3x^2 + 6x + 5$, prove that $f(1) = 9, f(3) = 23, f(-1) = -5$.

11. If $f(x, y) \equiv x^2 + 2xy + y^2$, find $f(y, x)$ and $f(x, x)$.

12. If $f(x) \equiv \sqrt{1-x^2}$, find $f(\sin x)$ and $f(\cos x)$.

13. If $f(x) \equiv x^2 + 4$ and $F(x) \equiv (x-1)$, find $f(F(x))$, $F(f(x))$, $f(f(x))$, $F(F(x))$.

14. If $f(x, y) \equiv \dfrac{x-y}{x+y}$, find $f(x, y) + f(y, x)$.

15. If $f(x) \equiv \dfrac{x-1}{x+1}$, prove that $\dfrac{f(x) - f(y)}{1 + f(x)f(y)} = \dfrac{x-y}{1+xy}$.

16. If $\phi(x) \equiv \sqrt{1-x^2}$, find $\phi(\sqrt{1-x^2})$.

17. If $f(x) \equiv \tan x$, find $\dfrac{f(x) + f(y)}{1 - f(x)f(y)}$. *Ans.* $\tan(x + y)$.

18. If $f(x) \equiv \sin x$ and $\phi(x) \equiv \cos x$, find $\phi(x)\phi(y) \mp f(x)f(y)$. *Ans.* $\cos(x \pm y)$.

19. If $f(x) \equiv \sin x$ and $\phi(x) \equiv \cos x$, find $f(x)\phi(y) \pm \phi(x)f(y)$. *Ans.* $\sin(x \pm y)$.

20. If $F(x) \equiv \log_{10} x$, prove that $F(x) + F(y) = F(xy)$ and $F(x^n) = nF(x)$.

21. If $\phi(x) \equiv a^x$, prove that $\phi(x)\phi(y) = \phi(x+y)$, $\phi(x) \div \phi(y) = \phi(x-y)$ and $\phi(nx) = \{\phi(x)\}^n$.

22. If $F(x) \equiv \cos 2x$, prove that

$$F(x) + F(y) = 2F\left(\frac{x+y}{2}\right)F\left(\frac{x-y}{2}\right).$$

23. If $f(x, y) \equiv Ax + By + C$, show that $f(x, y) = 0$ and $f(-y, x) = 0$ are the equations of two straight lines perpendicular to each other.

24. If $f(x, y) \equiv 2x + 3y - 4$, what do $f(x^2, y) = 0$, $f(x, y^2) = 0$, $f(x^2, y^2) = 0$, $f(x^2, 0) = 0$ represent? Plot the curves.

25. Find

$$|x^2 - x| \text{ when } x = \tfrac{1}{2}.$$
$$|\sin x| \text{ when } x = 210°.$$
$$|\log_{10} x| \text{ when } x = .001.$$

26. Find

$$|x^3 - x^2 + x| \text{ when } x = -a.$$
$$|x^4 + x^2 - x| \text{ when } x = -a.$$

27. Plot the curve $y = \pm\sqrt{1-(x-1)^2}$. Indicate which part of the curve is found by taking $y = +\sqrt{1-(x-1)^2}$, and which by taking $y = -\sqrt{1-(x-1)^2}$.

28. Plot the curve $y = 1 \pm \sqrt{2-(x-3)^2}$. Indicate which part of the curve is found by taking $y = 1 + \sqrt{2-(x-3)^2}$, and which by taking $y = 1 - \sqrt{2-(x-3)^2}$.

CHAPTER I

LIMITS

16. Definitions. To determine whether one number is **greater** or **less** than another, represent the two numbers as lengths from the origin on a straight line. The one represented by the length whose extremity lies farther to the right is the greater of the two.

For example, to determine whether -2 is greater or less than -10, represent the two numbers as lengths from the origin on a straight line (Fig. 4). The extremity of the line that represents -2 lies farther to the right. Then -2 is greater than -10.

FIG. 4.

A variable or function is said to **continually increase** in value when it assumes a succession of values each of which is **greater** than the preceding value.

A variable or function is said to **continually decrease** in value when it assumes a succession of values each of which is **less** than the preceding value.

17. Let S_n denote the sum of the first n terms of the series:

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots.$$

Then
$$S_n=1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}$$

$$=1\cdot\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}$$

$$=2-2\left(\frac{1}{2}\right)^n=2-\frac{1}{2^{n-1}}.$$

Since n denotes the number of terms taken, it must be a positive integer, and since any number of terms may be taken, it may be any positive integer. Start with a value of n, say $n=1$. Let n continually increase, and see what change is being produced in S_n.

By taking n large enough, $\frac{1}{2^{n-1}}$ can be made less than any positive number, δ, we may choose to name, no matter how small a number is chosen for δ.

For example, if δ is chosen to be $\frac{1}{79}$, $\frac{1}{2^{n-1}}$ can be made less than $\frac{1}{79}$ by taking n equal to 8.

Also, $\frac{1}{2^{n-1}}$ remains less than δ for all values of n subsequent to the one chosen to make it less than δ. Then $S_n = 2 - \frac{1}{2^{n-1}}$, where $\frac{1}{2^{n-1}}$ can be made to come as near the value zero as we please by taking n large enough, and where $\frac{1}{2^{n-1}}$ remains at least that near zero for all subsequent values of n. Then with regard to S_n, as n is allowed to continually increase, we may state:

(1) That S_n can be made to come as near the value 2 as we please by taking n large enough.

(2) That S_n remains at least that near 2 for all subsequent values of n.

18. That S_n, in the preceding article, comes and remains as near 2 as we please can be seen more clearly perhaps by representing its successive values as lengths from the origin on a straight line.

FIG. 5.

Let $A'OA$ be a line of indefinite length (see Fig. 5). On $A'OA$, plot the line OB, 2 units in length. Beginning with $n=1$, plot the successive values of S_n as lengths on $A'OA$. S_1, or S_n when $n=1$, $=2-1=1$. Therefore the extremity of S_1 is at A_1, which bisects OB. S_2, or S_n when

$n = 2$, $= 2 - \frac{1}{2}$. Therefore the extremity of S_2 is at A_2, which bisects A_1B. S_3, or S_n when $n = 3$, $= 2 - \frac{1}{4}$. Therefore the extremity of S_3 is at A_3, which bisects A_2B. The law which the successive values of S_n obey is now evident. It is: the extremity of S_n is found by bisecting the line $A_{n-1}B$ where A_{n-1} is the extremity of S_{n-1}.

Since the line with its extremity at B is continually bisected, it follows that if any distance BC (equal to δ, suppose), no matter how short, is marked off from B to the left, the extremity of S_n will eventually fall between C and B, and remain between C and B for all subsequent values of n.

19. Definitions. If a variable is allowed to **continually increase, becoming greater than any number we may choose to name, no matter how great that number may be,** it is said to **increase without limit** or to **become infinite.**

Thus, n in the problem of Art. 18 increases without limit or becomes infinite.

The notation used to indicate that a variable increases without limit or becomes infinite is $n = \infty$, which is read, "increases without limit" or "becomes infinite."

If, as the variable changes according to a given law, the function **comes** as near a fixed number as we please and **remains at least that near** for all subsequent values of the variable, the function is said to approach the fixed number as a **limit** as the variable changes according to the given law.

Thus, as n increases without limit, S_n, in the series $1 + \dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots$, comes as near 2 as we please, and remains at least that near 2 for all subsequent values of n. Then S_n is said to approach 2 as its limit as n increases without limit.

The notation used to indicate that as n increases without limit S_n approaches 2 as its limit is $\underset{n=\infty}{\text{limit}}\Big[S_n\Big] = 2$, which is read, "the limit of S_n as n increases without limit is 2."

20. In the example $S_n = 1 + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$, S_n is always less than its limit, and approaches it by continually increasing to it. That this is not always the case may be seen by examples.

Example 1. Find the limit which the sum of the first n terms of the series

$$-1 - \frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} - \cdots$$

approaches as n increases without limit.

$$S_n = -1 - \frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} - \cdots - \frac{1}{2^{n-1}}$$

$$= -1 \cdot \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}$$

$$= -2 + \frac{1}{2^{n-1}}.$$

Therefore $\underset{n=\infty}{\text{limit}} \left[S_n\right] = -2.$

In this example S_n is always greater than its limit and continually decreases to it.

Example 2. Find the limit which the sum of the first n terms of the series

$$1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$$

approaches as n increases without limit.

$$S_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots + (-1)^{n-1}\frac{1}{2^{n-1}}$$

$$= 1 \cdot \frac{1 - \left(-\frac{1}{2}\right)^n}{1 - \left(-\frac{1}{2}\right)}$$

$$= \frac{2}{3} - \frac{2}{3}\left(-\frac{1}{2}\right)^n = \frac{2}{3} + \frac{(-1)^{n+1}}{3 \cdot 2^{n-1}}.$$

Therefore $$\operatorname*{limit}_{n=\infty}\left[S_n\right]=\frac{2}{3}.$$

In this example S_n is alternately greater and less than its limit.

21. It must be carefully noticed that it is as essential that the function **remain** as near the fixed number as we please as that it can be made to **come** as near the fixed number as we please.

Consider the series

$$2-\frac{1}{2}+\frac{5}{4}-\frac{7}{8}+\cdots,$$

found by adding $+1$ and -1 alternately to the terms of the series

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots.$$

Let S_n denote the sum of the first n terms of the series.

Then $$S_n=2-\frac{1}{2}+\frac{5}{4}-\frac{7}{8}+\cdots+\left(\frac{1}{2^{n-1}}\pm 1\right),$$

where the plus sign must be taken in the nth term if n be odd, and the minus sign if n be even.

This series can be summed as follows:

$$S_n=(1+1)+\left(\frac{1}{2}-1\right)+\left(\frac{1}{2^2}+1\right)+\left(\frac{1}{2^3}-1\right)+\cdots+\left(\frac{1}{2^{n-1}}\pm 1\right)$$

$$=1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}+(1-1+1-1\cdots\pm 1),$$

by removing brackets and rearranging terms.

$$\therefore\ S_n=\frac{1-\dfrac{1}{2^n}}{1-\dfrac{1}{2}}+(0 \text{ or } 1, \text{ according as } n \text{ is even or odd}).$$

$$=2-\frac{1}{2^{n-1}}, \text{ if } n \text{ is even,}$$

or $$=3-\frac{1}{2^{n-1}}, \text{ if } n \text{ is odd.}$$

By taking n large enough and even, S_n can be made to come as near 2 as we please, and by taking n large enough and odd, S_n can be made to come as near 3 as we please. Then if we start with n any positive integer and allow it to increase by unity each time, S_n can be made to come as near 2 or 3 as we please. It does not, however, remain either as near 2 or 3 as we please for all subsequent values of n. It therefore approaches neither 2 nor 3 as a limit, and S_n as n increases without limit, by unity each time, approaches no limit.

22. Although the function considered in the preceding article failed to approach a limit, it always remained between two numbers. For example, it remained between 1 and 3. Now a function may fail to approach a limit in other ways, as the following examples will show:

Example 1. Consider the series

$$1 + 2 + 2^2 + 2^3 + \cdots.$$

$$S_n = 1 + 2 + 2^2 + 2^3 + \cdots + 2^{n-1}$$

$$= \frac{1 - 2^n}{1 - 2} = 2^n - 1.$$

As n increases without limit, S_n becomes greater than any number we may choose to name. It also remains greater for all subsequent values of n. Therefore S_n does not come and remain as near any given number as we please and therefore does not approach a limit.

Definition. If, as the variable changes according to a given law, the function becomes and remains **greater than any number we may choose to name, no matter how great that number may be,** it is said to **increase without limit** or **to become infinite.**

The notation used to indicate that as n increases without limit S_n increases without limit is $\underset{n=\infty}{\text{limit}}\left[S_n\right] = \infty$, which is read, "as n increases without limit, S_n increases without limit."

In Example 1, S_n increases without limit.

EXAMPLE 2. Consider the series

$$-1-2-2^2-2^3-\cdots.$$

$$S_n=-1-2-2^2-2^3-\cdots-2^n$$

$$=-1\cdot\frac{1-2^n}{1-2}=1-2^n.$$

As n increases without limit, S_n becomes less than any negative number we may choose to name no matter how great in absolute value that number may be. It also remains less for all subsequent values of n. Therefore S_n does not come and remain as near any given number as we please, and therefore does not approach a limit.

Definition. If, as the variable changes according to a given law, the function **becomes and remains less than any negative number we may choose to name, no matter how great in absolute value that number may be,** it is said to **decrease without limit** or to **become infinite negatively.**

The notation used to indicate that as n increases without limit S_n decreases without limit is $\underset{n=\infty}{\text{limit}}\Big[S_n\Big]=-\infty$, which is read, "as n increases without limit, S_n decreases without limit."

In Example 2, S_n decreases without limit.

EXAMPLE 3. Consider the series

$$1-2+2^2-2^3+\cdots.$$

$$S_n=1-2+2^2-2^3+\cdots+(-1)^{n-1}2^{n-1}$$

$$=\frac{1+(-1)^{n+1}2^n}{1+2}=\frac{(-1)^n2^n-1}{3}.$$

As n increases without limit, S_n is such that in absolute value it becomes greater than any number we may choose to name. It also remains greater in absolute value for all subsequent values of n. Therefore S_n does not come and remain as near any given number as we please and therefore does not approach a limit.

In this case S_n is such that in absolute value it increases without limit.

The notation used to indicate that as n increases without limit, S_n, in absolute value, increases without limit is $\underset{n=\infty}{\text{limit}}\left[|S_n|\right]=\infty$, which is read, "as n increases without limit, S_n, in absolute value, increases without limit."

EXERCISES

1. In the examples of Arts. 20, 21, and 22, represent the successive values of S_n as lengths from the origin on a straight line, determining the law which these values obey.

In each of the following series find the limit, if any, which S_n approaches as n increases without limit. In each case represent the successive values of S_n as lengths from the origin on a straight line.

2. $1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots.$

3. $1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots.$

4. $1-\frac{1}{4}+\frac{1}{4^2}-\frac{1}{4^3}+\cdots.$

5. $\frac{1}{2}-\left(\frac{1}{2}\right)^4+\left(\frac{1}{2}\right)^7-\left(\frac{1}{2}\right)^{10}+\cdots.$

6. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots.$

SUGGESTION.

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{n(n-1)}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)$$
$$+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n-1}\right).$$

7. $2+\frac{3}{2}+\frac{5}{4}+\frac{9}{8}+\frac{17}{16}+\cdots.$

8. $2+\frac{1}{2}+\frac{5}{4}+\frac{7}{8}+\cdots.$

23. Definitions. If a variable is allowed to continually decrease becoming less than any negative number we may choose to name, no matter how great in absolute value that number may be, it is said to decrease without limit or to become infinite negatively.

The notation used to indicate that a variable decreases without limit or becomes infinite negatively is $-\infty$, which is read, "decreases without limit or becomes infinite negatively."

If a variable assumes a succession of values each of which is nearer a given number than the preceding value, and is made to differ from the given number by a value as small as we please, it is said to approach the given number as a limit.

The notation used to indicate that a variable approaches a given number a is $\doteq a$ which is read, "approaches a as a limit."

NOTE. Attention should be called to the notation $\doteq$ and $=$. When the variable approaches a given number, we use the notation $\doteq$, and when it increases or decreases without limit, we use the notation $=$.

24. In all problems in limits we shall suppose that the variable changes according to one or other of the following laws:

1st. It increases without limit.
2d. It decreases without limit.
3d. It approaches a given number as a limit.

When the variable changes according to one of these laws, it may assume an unlimited number of successions of values, the only restriction being that the values it assumes are in accordance with this law.

Thus, if the variable increases without limit, it may assume an unlimited number of successions of values, the only restriction being that, in any succession, each value is greater than the preceding and that eventually the values become greater than any number we may choose to name.

For example, it may assume any one of the successions:

	-1,	$2\frac{1}{2}$,	6,	$9\frac{1}{2}$,	13,	$16\frac{1}{2}$,	$\cdots$,
	$\frac{1}{2}$,	$\frac{3}{4}$,	1,	$1\frac{1}{4}$,	$1\frac{1}{2}$,	$1\frac{3}{4}$,	$\cdots$,
	1,	2,	3,	4,	5,	6,	$\cdots$,
	1,	3,	5,	7,	9,	11,	$\cdots$,
or	1,	2,	4,	8,	16,	32,	$\cdots$,

If in addition to the given law the variable is further restricted, some of these successions become no longer possible. Thus, for example, if the variable denotes the number of terms taken in a series, the first two of the above successions become no longer possible.

If the function approaches a limit when the variable changes according to any one of the above laws, and is not otherwise restricted, it will approach the same limit when the variable is further restricted. This is obvious, because if the function approaches a limit for any succession of values subject to a given law, it will also approach the same limit when the variable is limited to fewer of these successions.

In the theorems of Arts. 26, 27, and 28, the variable is supposed to change according to one or other of the above laws, and is not otherwise restricted. By the preceding paragraph, the theorems will also be true if the variable be further restricted.

As an example in which the function approaches a limit as the variable approaches a given number, consider the function $x+1$ as x approaches 2.

By giving x a value near enough to 2, $x+1$ can be made to come as near the value 3 as we please. Also, $x+1$ remains at least that near 3 for all subsequent values of x. Therefore $\underset{x \doteq 2}{\text{limit}}\Big[x+1\Big]=3.$

To illustrate by points on a line, choose some succession of values, suppose 1, $2\frac{1}{2}$, $1\frac{3}{4}$, $2\frac{1}{8}$, $\cdots$.

As x takes each of these values in succession, $x+1$ takes each of the values 2, $3\frac{1}{2}$, $2\frac{3}{4}$, $3\frac{1}{8}$, $\cdots$ in succession (see Fig. 6). If we mark off a length of line CB (equal to δ), no matter how small, about the point which is three units from the origin, $x+1$ will eventually take a value such that the extremity of the line that represents it comes and remains in the region CB.

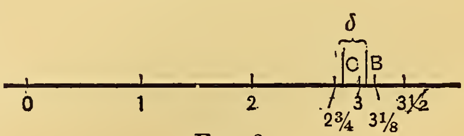


FIG. 6.

25. **Definition.** An **infinitesimal** is a function that can be made to approach **zero** as a limit.

EXAMPLE 1. In the example of Art. 17, $S_n = 2 - \frac{1}{2^{n-1}}$. Since $\underset{n=\infty}{\text{limit}}\left[\frac{1}{2^{n-1}}\right] = 0$, $-\frac{1}{2^{n-1}}$ is infinitesimal as n becomes infinite, being integral.

EXAMPLE 2. Since $\underset{x \doteq 0}{\text{limit}}\left[x - 1\right] = 0$, $x - 1$ is infinitesimal as x approaches 1.

26. From the definition of a limit, the difference between a function and its limit is infinitesimal.

EXAMPLE 1. In $S_n = 2 - \frac{1}{2^{n-1}}$, $-\frac{1}{2^{n-1}}$ is the difference between S_n and its limit 2, and is therefore infinitesimal as n increases without limit.

EXAMPLE 2. Since $\underset{x \doteq 2}{\text{limit}}\left[x + 1\right] = 3$, the difference between $x + 1$ and its limit 3 is infinitesimal as x approaches 2.

27. If the difference between a function and a constant is infinitesimal, the constant is the limit of the function.

For, let U be a function and a a constant such that $U - a = \epsilon$ where ϵ is infinitesimal. Then $U = a + \epsilon$. Since ϵ is infinitesimal, U can be made to come and remain as near a as we please. Therefore U approaches a as its limit.

This theorem is the converse of that of the preceding article.

28. The following theorems are necessary to the subsequent investigation:

Theorem I. Any finite multiple of an infinitesimal is an infinitesimal.

Let ϵ be an infinitesimal and m any number. To prove that $m\epsilon$ is infinitesimal.

Proof. Choose any number ϵ' as small as we please. Allow ϵ to become less than $\frac{\epsilon'}{m}$. Then $m\epsilon$ is less than ϵ'. Since ϵ is

infinitesimal, it remains less than $\frac{\epsilon'}{m}$ for all subsequent values of the variable. Therefore $m\epsilon$ can be made to become and remain less than ϵ'. Therefore $m\epsilon$ can be made to become and remain as small as we please. It is therefore infinitesimal.

EXAMPLE. If $\epsilon = \frac{1}{2^{n-1}}$ is infinitesimal, $1000\left(\frac{1}{2^{n-1}}\right)$ is infinitesimal.

A special case of this theorem is the following:

A fraction whose numerator is infinitesimal, and whose denominator is any number not zero, is infinitesimal.

EXAMPLE. Since $\underset{x \doteq 1}{\text{limit}}\left[x-1\right]=0$, $\therefore \frac{x-1}{7}$ is infinitesimal as x approaches 1.

Theorem II. The sum of a finite number of infinitesimals is infinitesimal.

Let ϵ_1, ϵ_2, ϵ_3, $\cdots$, ϵ_n be n infinitesimals. To prove that $\Sigma = \epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots + \epsilon_n$ is infinitesimal.

Proof. Since each of the infinitesimals may be made to become and remain as small as we please, let us make each to become and remain smaller in absolute value than some positive infinitesimal ϵ. Then Σ becomes and remains less than $n\epsilon$. But $n\epsilon$ is infinitesimal, by Theorem I. Therefore Σ is infinitesimal.

NOTE. Attention must be called to the necessity for the word *finite* in the above theorems. Suppose that a line of given length is divided into n equal parts by bisecting the line and continually bisecting each part. If n is allowed to increase without limit, each part is infinitesimal. The sum of the infinitesimals, however, is not infinitesimal. It is the length of the given line. In this case there is not a finite number of infinitesimals, but a number that increases without limit as each part approaches zero as its limit.

Theorem III. If two functions differ only by an infinitesimal, and one of them approaches a limit, the other approaches the same limit.

Let U_1 and U_2 be two functions differing only by an infinitesimal, and suppose that U_1 approaches a as its limit. To prove that U_2 also approaches a as its limit.

Proof. By supposition, $U_1 - U_2 = \epsilon$, where ϵ is infinitesimal.

By the definition of a limit, $U_1 - a = \epsilon'$, where ϵ' is infinitesimal.

Subtract. $\therefore U_2 - a = \epsilon' - \epsilon.$

Since ϵ' and ϵ are infinitesimals, $\epsilon' - \epsilon$, by Theorem II, is infinitesimal.

Therefore, by the theorem of Art. 27, U_2 approaches a as its limit.

Theorem IV. If each of a finite number of functions approaches a limit, the limit of the sum of the functions is the sum of their respective limits.

Let $U_1, U_2, U_3, \cdots, U_n$ be n functions which approach the limits $a_1, a_2, a_3, \cdots, a_n$, respectively. To prove that the limit of $U_1 + U_2 + U_3 + \cdots + U_n$ is $a_1 + a_2 + a_3 + \cdots + a_n$.

Proof. By supposition,

$$U_1 = a_1 + \epsilon_1,$$
$$U_2 = a_2 + \epsilon_2,$$
$$U_3 = a_3 + \epsilon_3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$U_n = a_n + \epsilon_n,$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \cdots, \epsilon_n$ are infinitesimals.

$$\therefore U_1 + U_2 + U_3 + \cdots + U_n = (a_1 + a_2 + a_3 + \cdots + a_n) + (\epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots \epsilon_n).$$

Now $\epsilon_1 + \epsilon_2 + \epsilon_3 + \cdots + \epsilon_n$ is infinitesimal, by Theorem II.

Therefore $U_1 + U_2 + U_3 + \cdots + U_n$ approaches $a_1 + a_2 + a_3 + \cdots + a_n$ as a limit, by the theorem of Art. 27.

EXAMPLE. $\underset{x \doteq 2}{\text{limit}} \left[x^3 - 3 \right] = \underset{x \doteq 2}{\text{limit}} \left[x^3 \right] + \underset{x \doteq 2}{\text{limit}} \left[-3 \right].$

Theorem V. If each of a finite number of functions approaches a limit, the limit of the product of these functions is the product of their respective limits.

Let $U_1, U_2, U_3, \cdots, U_n$ be n functions which approach the limits $a_1, a_2, a_3, \cdots, a_n$ respectively. To prove that the limit of $U_1 \cdot U_2 \cdot U_3 \cdots U_n$ is $a_1 \cdot a_2 \cdot a_3 \cdots a_n$.

Proof. By supposition,

$$U_1 = a_1 + \epsilon_1,$$
$$U_2 = a_2 + \epsilon_2,$$
$$U_3 = a_3 + \epsilon_3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$U_n = a_n + \epsilon_n,$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \cdots, \epsilon_n$ are infinitesimals.

$\therefore$ $U_1 \cdot U_2 \cdot U_3 \cdots U_n = a_1 \cdot a_2 \cdot a_3 \cdots a_n$ plus a finite number of terms, each containing an infinitesimal factor. Each of these factors is infinitesimal, and therefore, by Theorem II, their sum is infinitesimal. Therefore, by the theorem of Art. 27, $U_1 \cdot U_2 \cdot U_3 \cdots U_n$ approaches $a_1 \cdot a_2 \cdot a_3 \cdots a_n$ as a limit.

EXAMPLE. $\underset{x \doteq 3}{\text{limit}} \left[x^3 \cdot x^2 \right] = \underset{x \doteq 3}{\text{limit}} \left[x^3 \right] \cdot \underset{x \doteq 3}{\text{limit}} \left[x^2 \right].$

Theorem VI. If the numerator of a fraction approaches the limit zero and the denominator approaches a limit other than zero, the fraction approaches zero as its limit.

Let $\dfrac{U}{V}$ be a fraction in which the limit of U is 0 and the limit of V is b, where b is not zero. To prove that $\dfrac{U}{V}$ approaches zero as its limit.

Proof. Since V approaches the limit b, $|V|$ can be made to become and remain larger than some positive number c. Then for all subsequent values of the variable, $\left|\dfrac{U}{V}\right|$ is less than $\dfrac{|U|}{c}$. But $\dfrac{|U|}{c}$ approaches zero as its limit. Therefore $\left|\dfrac{U}{V}\right|$ approaches zero as its limit. Therefore $\dfrac{U}{V}$ approaches zero as its limit.

EXAMPLE. $\underset{x \doteq 1}{\text{limit}} \left[\dfrac{x-1}{x+2} \right] = 0.$

Theorem VII. If the numerator of a fraction approaches a limit, and the denominator approaches a limit other than zero, the limit of the fraction is the limit of the numerator divided by the limit of the denominator.

Let $\frac{U}{V}$ be a fraction in which the limit of U is a, and the limit of V is b, where b is not zero. To prove that $\frac{U}{V}$ approaches the limit $\frac{a}{b}$.

Proof. $U = a + \epsilon,$

$V = b + \beta$, where ϵ and β are infinitesimals.

$$\therefore \quad \frac{U}{V} = \frac{a+\epsilon}{b+\beta}$$

$$= \frac{a}{b} - \frac{\frac{a}{b}\beta - \epsilon}{b+\beta}.$$

Now $\dfrac{\frac{a}{b}\beta - \epsilon}{b+\beta}$ approaches zero as its limit, by Theorem VI.

Therefore $\frac{U}{V}$ approaches $\frac{a}{b}$ as its limit.

EXAMPLE. $\displaystyle \lim_{x \doteq -2}\left[\frac{x-3}{x-5}\right] = \frac{\displaystyle\lim_{x \doteq -2}\left[x-3\right]}{\displaystyle\lim_{x \doteq -2}\left[x-5\right]} = \frac{-5}{-7} = \frac{5}{7}.$

As illustrations of the manner of applying the above theorems, consider the following examples:

EXAMPLE 1. Find $\displaystyle\lim_{x \doteq a}\left[x^n\right]$, where n is a positive integer.

$$x^n = x \cdot x \cdot x \cdot \cdots \text{ to } n \text{ factors.}$$

$$\therefore \lim_{x \doteq a}\left[x^n\right] = \lim_{x \doteq a}\left[x\right] \cdot \lim_{x \doteq a}\left[x\right] \cdot \lim_{x \doteq a}\left[x\right] \cdots$$

to n factors, by Theorem V,

$$= a \cdot a \cdot a \cdot \cdots \text{ to } n \text{ factors,}$$

$$= a^n.$$

EXAMPLE 2. Find $\lim_{x \doteq 1}\left[\dfrac{3x^2+x-1}{x^3-3x+4}\right]$.

$$\lim_{x \doteq 1}\left[x^3-3x+4\right]=\lim_{x \doteq 1}\left[x^3\right]+\lim_{x \doteq 1}\left[-3x\right]+\lim_{x \doteq 1}\left[4\right],$$

by Theorem IV,

$$=1-3+4$$
$$=2.$$

Therefore, since the limit of the denominator is not zero,

$$\lim_{x \doteq 1}\left[\frac{3x^2+x-1}{x^3-3x+4}\right]=\frac{\lim_{x \doteq 1}\left[3x^2+x-1\right]}{\lim_{x \doteq 1}\left[x^3-3x+4\right]}, \text{ by Theorem VII,}$$
$$=\frac{3}{2}.$$

EXAMPLE 3. Find $\lim_{x=\infty}\left[\dfrac{2x^2-x+1}{x^2+3x-2}\right]$.

Divide both numerator and denominator by x raised to the highest power to which it appears in the fraction, *i.e.* by x^2.

$$\therefore \lim_{x=\infty}\left[\frac{2x^2-x+1}{x^2+3x-2}\right]=\lim_{x=\infty}\left[\frac{2-\frac{1}{x}+\frac{1}{x^2}}{1+\frac{3}{x}-\frac{2}{x^2}}\right]=2.$$

EXERCISES

1. Prove that if the numerator of a fraction does not approach zero, and the denominator does approach zero, the fraction either (1) increases without limit, or (2) decreases without limit, or (3) is such that in absolute value it increases without limit.

2. Prove that if a function (1) increases without limit, (2) decreases without limit, any finite multiple of the function will (1) increase without limit, (2) decrease without limit.

Choose $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$ as a succession of values for x, and for the first six values, plot as lengths from the origin on a straight

line, the values which the function in each of exercises 3 and 4 assumes. In each case, what is the length of the region determined by the sixth value of x, within which the extremities of the lengths representing the function must lie for all subsequent values of x?

3. $\dfrac{x-1}{7}$.

4. $\dfrac{3x+1}{x-6}$.

Choose 1, $2\frac{1}{2}$, $1\frac{3}{4}$, $2\frac{1}{8}$, $\cdots$ as the succession of values for x, and give a treatment similar to that in exercises 3 and 4 for the function in each of the exercises 5 and 6.

5. $\dfrac{2x+1}{x-3}$.

6. $\dfrac{x-3}{x+1}$.

For any succession of values for x, in each of the following exercises, find:

7. $\underset{x \doteq 0}{\text{limit}}\left[\dfrac{3}{x}\right]$.

8. $\underset{x \doteq 1}{\text{limit}}\left[\dfrac{3x^2-2x+4}{x^2-2x+1}\right]$.

9. $\underset{x \doteq 0}{\text{limit}}\left[\dfrac{(3x-2)(6x+7)}{(3x+4)(x^2-1)}\right]$.

10. $\underset{x \doteq 1}{\text{limit}}\left[\dfrac{(3x-2)(6x+7)}{(3x+4)(x^2-1)}\right]$.

11. $\underset{x \doteq a}{\text{limit}}\left[\dfrac{x^2+ax+a^2}{a^2+x^2}\right]$.

12. $\underset{x = \infty}{\text{limit}}\left[\dfrac{x^3+x^2+1}{3x^3+x-4}\right]$.

13. $\underset{x = -\infty}{\text{limit}}\left[\dfrac{x^2+1}{x^3-1}\right]$.

14. $\underset{x = \infty}{\text{limit}}\left[\dfrac{x^3+1}{2x^2-1}\right]$.

15. $\underset{x \doteq 0}{\text{limit}}\left[\dfrac{\sin x}{x-1}\right]$.

16. $\underset{x \doteq 0}{\text{limit}}\left[\dfrac{\cos x}{x}\right]$.

17. $\underset{x \doteq \frac{\pi}{2}}{\text{limit}}\left[\dfrac{x}{\tan x}\right]$.

18. $\underset{x \doteq \frac{\pi}{2}}{\text{limit}}\left[\cot x + \tan x\right]$.

29. **Function infinite or infinite negatively for a value of the variable.** Let $f(x)$ be a single-valued function of the variable x. The function $f(x)$ is said to be (1) infinite, (2) infinite negatively for a value x_0 of x, if as x approaches x_0, $f(x)$ becomes (1) greater, (2) less than any number we may choose to name

(see Art. 22). The curve $y=f(x)$ in rectangular coördinates, therefore, will be (1) infinite, (2) infinite negatively for a value x_0 of x, if as x approaches x_0 the ordinate of the curve becomes and remains (1) greater, (2) less than any length we may choose to name.

It frequently happens that a function is infinite or infinite negatively for a value x_0 as x approaches x_0 being always greater than it, and the opposite, namely infinite negatively or infinite, as x approaches x_0 being always less than it.

Thus, the function $\dfrac{x}{x-2}$ when $x=2$, is infinite if x approaches 2 being always greater than 2, and is infinite negatively if x approaches 2 being always less than 2.

In the neighborhood of $x=2$, the curve $y=\dfrac{x}{x-2}$ is as in Fig. 7.

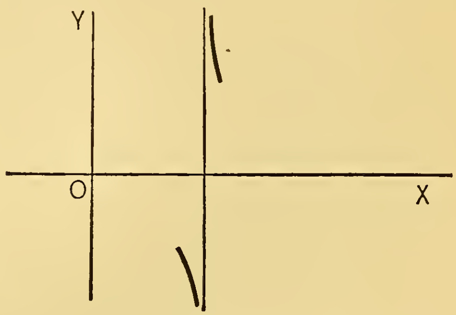


FIG. 7.

30. Function finite between two values of the variable. Let $f(x)$ be a single-valued function of x. The function $f(x)$ is said to be finite between two values x_1 and x_2 of x, if there is no value x_0 of x between x_1 and x_2, for which $f(x)$ does not have a definite value.

Thus, the function $\dfrac{x}{x-2}$ is finite between any two values of x which do not contain the value $x=2$ between them.

31. Function continuous or discontinuous for a value of the variable. Let $f(x)$ be a single-valued function of x. Suppose that, when x has a value near a particular value x_0, $f(x)$ is finite. Let x_0-h and x_0+h be two arbitrarily chosen values of x near x_0. The function $f(x)$ is said to be continuous for $x=x_0$,

if $$\underset{h\doteq 0}{\text{limit}}\Big[f(x_0-h)\Big]=\underset{h\doteq 0}{\text{limit}}\Big[f(x_0+h)\Big]=f(x_0).$$

The function $f(x)$ is said to be discontinuous for $x=x_0$, if

$$\lim_{h \doteq 0}\left[f(x_0-h)\right] \neq \lim_{h \doteq 0}\left[f(x_0+h)\right].$$

It is also discontinuous for $x=x_0$, if

$$\lim_{h \doteq 0}\left[f(x_0-h)\right] = \lim_{h \doteq 0}\left[f(x_0+h)\right] \neq f(x_0),$$

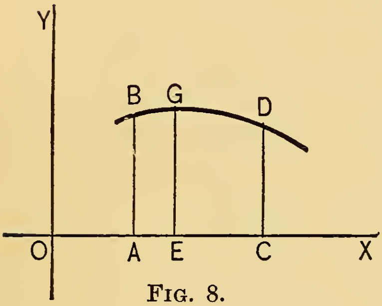


FIG. 8.

but this case is of rare occurrence.

In the curve $y=f(x)$ in rectangular coördinates, x_0-h, x_0 and x_0+h represent abscissas, and $f(x_0-h)$, $f(x_0)$, and $f(x_0+h)$, ordinates. In Fig. 8, $f(x_0-h)$ is AB, $f(x_0)$ is EG, and $f(x_0+h)$ is CD. If

$$\lim_{h \doteq 0}\left[f(x_0-h)\right] = \lim_{h \doteq 0}\left[f(x_0+h)\right] = f(x_0),$$

which is the case in this figure, the curve is continuous for $x=x_0$, or at the point E.

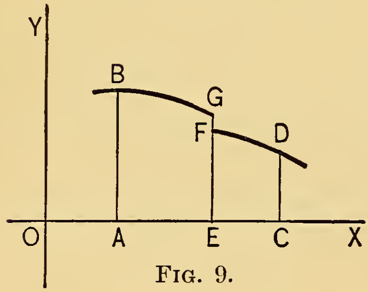


FIG. 9.

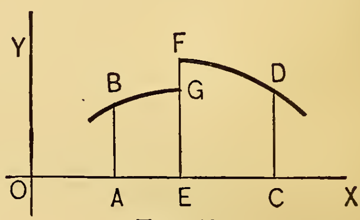


FIG. 10.

In Figs. 9 and 10, $f(x_0-h)$ is AB, and $f(x_0+h)$ is CD. If $\lim_{h \doteq 0}\left[f(x_0-h)\right] \neq \lim_{h \doteq 0}\left[f(x_0+h)\right]$, which is the case in either figure, because AB approaches the limit EG, and CD the limit EF, the curve is discontinuous for $x=x_0$, or at the point E.

So far the curve was supposed to remain finite for values of x near $x=x_0$.

The function $f(x)$ is also said to be discontinuous for a value $x = x_0$, if it becomes infinite or infinite negatively for the value $x = x_0$.

32. **Definitions.** When a function **remains finite** for a value of the variable and is **discontinuous for this value,** it is said to have a **finite discontinuity** for this value of the variable.

When a function becomes **infinite** or **infinite negatively** for a value of the variable, it is said to have an **infinite discontinuity** for this value of the variable.

33. All the functions with which the student is likely to meet are continuous for all values of the variable excepting those which make the function infinite or infinite negatively.

No attempt is made here to prove a function continuous, and it will be assumed in the following chapters that all the functions considered are continuous excepting for those values of the variable for which they become infinite or infinite negatively.

EXERCISES

1. Plot the curve $y = \dfrac{x(x-1)}{(x-2)(x-3)}$ for values of x, 1st, near 1; 2d, near 2; 3d, near 3.

2. Plot the curve $y = \dfrac{x+2}{(x-1)^2}$ for values of x near 1.

3. Show that $x(x-1)$ is continuous when $x = 1$. Plot the curve $y = x(x-1)$ for values of x near 1.

4. Is the function $\dfrac{1}{1+e^{\frac{1}{x}}}$ continuous, 1st, when $x = 1$; 2d, when $x = 0$? Plot the curve $y = \dfrac{1}{1+e^{\frac{1}{x}}}$ for values of x near these values.

5. Plot the curve $y = \dfrac{1}{x}\sin x$ for values of x near 0. What can be said about the curve when $x = 0$?

CHAPTER II

DERIVATIVES

34. Definitions. The increase in a variable due to its having passed from one value to another is called the increment of the variable.

The increase in the function due to the variable having passed from one value to another is called the increment of the function.

An increment is usually denoted by writing Δ before the variable, Δ being merely a symbol for the words *increment of.*

For example, in the equation $y = x^2$, suppose that the variable has passed from the value 6 to the value 8. Its increment is 2. That is, $\Delta x = 2$. When x passed from 6 to 8, y passed from 36 to 64. Its increment is $64 - 36$, or 28. That is, $\Delta y = 28$. Or again, suppose that x has passed from the value 6 to the value 4. Its increment is -2. That is, $\Delta x = -2$. When x passed from 6 to 4, y passed from 36 to 16. Its increment is $16 - 36$, or -20. That is, $\Delta y = -20$.

35. In the equation $y = x^2$, let x have the value 6, and calculate the values of Δy and $\frac{\Delta y}{\Delta x}$, found by giving Δx certain values. The values given to Δx and the values resulting for Δy and $\frac{\Delta y}{\Delta x}$ are given in the scheme at the top of the following page. From a study of this scheme, we see that, in this particular example, as Δx approaches zero, Δy approaches zero, but $\frac{\Delta y}{\Delta x}$ does not approach zero. It approaches 12.

If $\Delta x =$	then $\Delta y =$	and $\dfrac{\Delta y}{\Delta x} =$
3	45	15
2	28	14
1	13	13
.1	1.21	12.1
.01	.1201	12.01
.001	.012001	12.001
.0001	.00120001	12.0001
Δx	$12\,\Delta x + \overline{\Delta x}^2$	$12 + \Delta x$

36. Instead of giving x the value 6, as in the last article, we may give it any value x_0. Indicate by y_0 the value of x^2 when x has the value x_0. The increment of y is found by giving x the value $x_0 + \Delta x$, where Δx is an arbitrary value, and subtracting y_0 from the result. This increment by definition is Δy.

Then
$$\Delta y = (x_0 + \Delta x)^2 - x_0^2$$
$$= 2\,x_0 \Delta x + \overline{\Delta x}^2.$$

Divide by Δx. $\therefore \dfrac{\Delta y}{\Delta x} = 2\,x_0 + \Delta x.$

Pass to limits. $\therefore \underset{\Delta x \doteq 0}{\text{limit}} \left[\dfrac{\Delta y}{\Delta x}\right] = 2\,x_0.$

This gives $\underset{\Delta x \doteq 0}{\text{limit}} \left[\dfrac{\Delta y}{\Delta x}\right]$ for any value x_0 of x. In particular, if $x_0 = 6$, $\underset{\Delta x \doteq 0}{\text{limit}} \left[\dfrac{\Delta y}{\Delta x}\right] = 12$, the result of the last article.

Since x_0 is any value of x, we may drop the subscript and say that $\underset{\Delta x \doteq 0}{\text{limit}} \left[\dfrac{\Delta y}{\Delta x}\right] = 2\,x$ for any value of x.

37. **Definition.** In the equation $y = f(x)$, $\underset{\Delta x \doteq 0}{\text{limit}} \left[\dfrac{\Delta y}{\Delta x}\right]$, when x has some particular value x_0, is called the **derivative** of y with respect to x when $x = x_0$, or, sometimes, the **differential coefficient** of y with respect to x when $x = x_0$, and is written $\left.\dfrac{dy}{dx}\right|_{x=x_0}$.

Since x_0 is any value of x, we may drop the subscript and say, $\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta x}\right]$ for any value of x is $\left.\frac{dy}{dx}\right|_{x=x}$. This is written more briefly, $\frac{dy}{dx}$.

38. In an equation $y = f(x)$ the successive steps in finding $\frac{dy}{dx}$ are:

1st. Let x have the value x_0 and calculate the corresponding value y_0 of y.

2d. Let x have the value $x_0 + \Delta x$, where Δx is an arbitrary increment, and calculate the corresponding value of y. This value will be $y_0 +$ the increment in y due to the increment given to x, or $y_0 + \Delta y$.

3d. Subtract y_0 from $y_0 + \Delta y$.

4th. Divide Δy by Δx.

5th. Pass to limits.

The following examples will illustrate the method:

Example 1. Find $\left.\frac{dy}{dx}\right|_{x=5}$ when $y = x^3 - 2x + 4$.

Let $x = 5$. $\qquad \therefore\ y_0 = 119.$

Let $x = 5 + \Delta x$. $\qquad \therefore\ y_0 + \Delta y = (5 + \Delta x)^3 - 2(5 + \Delta x) + 4.$

Subtract.
$$\therefore\ \Delta y = (5 + \Delta x)^3 - 2(5 + \Delta x) + 4 - 119$$
$$= 73\,\Delta x + 15\,\overline{\Delta x}^2 + \overline{\Delta x}^3.$$

Divide by Δx. $\qquad \therefore\ \frac{\Delta y}{\Delta x} = 73 + 15\,\Delta x + \overline{\Delta x}^2.$

Pass to limits. $\qquad \therefore\ \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta x}\right] = 73$ when $x = 5$.

$$\therefore\ \left.\frac{dy}{dx}\right|_{x=5} = 73.$$

EXAMPLE 2. Find $\left.\dfrac{dy}{dx}\right|_{x=x_0}$ when $y = x^3 - 2x + 4$.

Let $x = x_0$. $\therefore\ y_0 = x_0^3 - 2x_0 + 4.$

Let $x = x_0 + \Delta x$. $\therefore\ y_0 + \Delta y = (x_0 + \Delta x)^3 - 2(x_0 + \Delta x) + 4.$

Subtract. $\therefore\ \Delta y = (x_0 + \Delta x)^3 - 2(x_0 + \Delta x) + 4 - (x_0^3 - 2x_0 + 4)$

$$= (3x_0^2 - 2)\Delta x + 3x_0\overline{\Delta x}^2 + \overline{\Delta x}^3.$$

Divide by Δx. $\therefore\ \dfrac{\Delta y}{\Delta x} = 3x_0^2 - 2 + 3x_0\Delta x + \overline{\Delta x}^2.$

Pass to limits. $\therefore\ \underset{\Delta x \doteq 0}{\text{limit}}\left[\dfrac{\Delta y}{\Delta x}\right] = 3x_0^2 - 2$ when $x = x_0$.

$$\therefore\ \left.\frac{dy}{dx}\right|_{x=x_0} = 3x_0^2 - 2.$$

EXAMPLE 3. Find $\dfrac{dy}{dx}$ when $y = \sqrt{x}$.

Let $x = x_0$. $\therefore\ y_0 = \sqrt{x_0}.$

Let $x = x_0 + \Delta x$. $\therefore\ y_0 + \Delta y = \sqrt{x_0 + \Delta x}.$

Subtract. $\therefore\ \Delta y = \sqrt{x_0 + \Delta x} - \sqrt{x_0}.$

Divide by Δx. $\therefore\ \dfrac{\Delta y}{\Delta x} = \dfrac{\sqrt{x_0 + \Delta x} - \sqrt{x_0}}{\Delta x}.$

If we take the limit of this expression as it stands, we encounter the indeterminate form $\frac{0}{0}$. The difficulty can be avoided by rationalizing the numerator. Multiply both numerator and denominator, therefore, by $\sqrt{x_0 + \Delta x} + \sqrt{x_0}$.

$$\therefore\ \frac{\Delta y}{\Delta x} = \frac{\Delta x}{\Delta x(\sqrt{x_0 + \Delta x} + \sqrt{x_0})} = \frac{1}{\sqrt{x_0 + \Delta x} + \sqrt{x_0}}.$$

Pass to limits. $\therefore\ \underset{\Delta x \doteq 0}{\text{limit}}\left[\dfrac{\Delta y}{\Delta x}\right] = \dfrac{1}{2\sqrt{x_0}}$ when $x = x_0$.

$$\therefore\ \frac{dy}{dx} = \frac{1}{2\sqrt{x}}.$$

EXERCISES

1. In the equation $y = x^2 - 2x$, calculate the values of Δy and $\frac{\Delta y}{\Delta x}$ corresponding to the values 4, 3, 2, 1, .1, .01, .001 of Δx, when $x = 5$. Find $\underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right]$ when $x = 5$.

2. In the equation $y = x^3$, calculate the values of Δy and $\frac{\Delta y}{\Delta x}$ corresponding to the values 3, 2, 1, .1, .01, .001 of Δx, when $x = 10$. Find $\underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right]$ when $x = 10$.

Find $\left.\frac{dy}{dx}\right|_{x=x_0}$ in each of the following cases:

3. $y = x^2 - 2x$, (*a*) when $x_0 = \frac{1}{2}$; (*b*) when $x_0 = 2$.

4. $y = (x-1)(3x+4)$, (*a*) when $x_0 = 3$; (*b*) when $x_0 = 4$.

5. $y = x^3$ when $x_0 = 3$.

6. $y = \frac{x-a}{x+a}$ when $x_0 = a$.

Find $\frac{dy}{dx}$ in each of the following cases:

7. $y = x^{\frac{3}{2}}$.

8. $y = \frac{2}{\sqrt{x+1}}$.

9. $y = \sqrt[4]{x}$.

10. $y = \frac{x+1}{x^2+1}$.

11. $y = \frac{x+2}{x^2-1}$.

12. $y = \frac{x^2-1}{x^2+1}$.

CHAPTER III

GEOMETRICAL INTERPRETATION OF A DERIVATIVE. PROBLEMS IN SPEED

39. We shall now consider the geometrical interpretation of a derivative.

Let $y = f(x)$ be the equation of a given curve. To avoid complications, we shall suppose that $f(x)$ is single valued and continuous. Since $f(x)$ is not known, we cannot plot the curve, but, as we wish a geometrical picture of the equation, we shall suppose that part of the curve at least, if it could be plotted, would be as drawn in Fig. 11. The student can readily satisfy himself that the same reasoning will hold for the curve drawn in any other position.

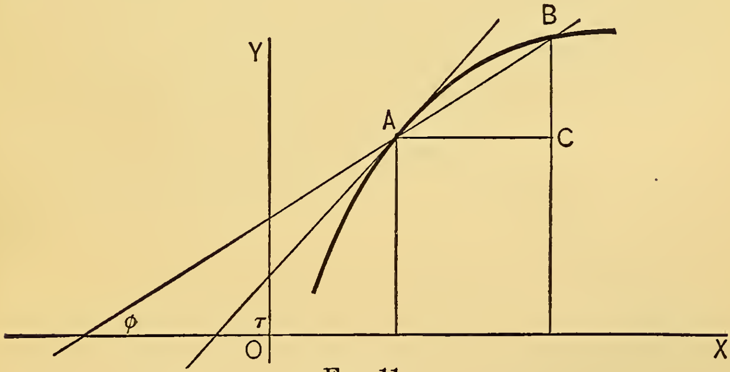


FIG. 11.

To let x assume a particular value x_0 in finding $\left.\dfrac{dy}{dx}\right|_{x=x_0}$ is geometrically to choose a length measured along the x-axis as the abscissa of a point on the curve. To calculate the corresponding value of y is geometrically to find the corresponding ordinate. Then the coördinates of A are (x_0, y_0). Similarly,

the coördinates of B are $(x_0+\Delta x, y_0+\Delta y)$. The result when y_0 is subtracted from $y_0+\Delta y$ is Δy. In the figure, Δy is CB. Now $\frac{\Delta y}{\Delta x}=\tan\phi$, where $\phi=\angle CAB$. Therefore $\frac{\Delta y}{\Delta x}$ is the slope of the secant line through the points A and B. As $\Delta x \doteq 0$, $B \doteq A$ and the points of intersection of the secant line approach coincidence. The limit of the secant line as B approaches A is by definition the tangent line to the curve at the point A. Let the angle which the tangent line at the point A makes with the x-axis be denoted by τ. As Δx approaches zero, ϕ approaches τ and

$$\operatorname*{limit}_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right]=\tan\tau.$$

Therefore $\left.\frac{dy}{dx}\right|_{x=x_0}$ is geometrically the slope of the tangent line to the curve $y=f(x)$ at the point (x_0, y_0) on the curve.

40. As an illustration consider the geometrical interpretation of $\left.\frac{dy}{dx}\right|_{x=x_0}$ in the parabola $y^2=4x$.

To put the equation in the form $y=f(x)$, solve for y. $\therefore\ y=\pm 2\sqrt{x}$. This is a double-valued function of x (see Art. 14). We shall here consider only $y=+2\sqrt{x}$ and confine our attention to the part of the curve above the x-axis.

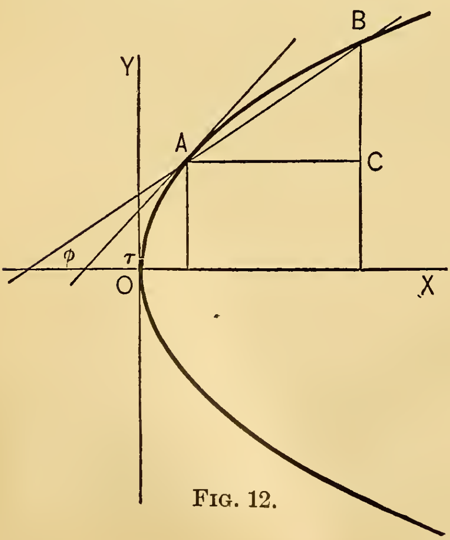


Fig. 12.

Let $x=x_0$.

$$\therefore\ y_0=2\sqrt{x_0}.$$

Let $x=x_0+\Delta x$.

$$\therefore\ y_0+\Delta y=2\sqrt{x_0+\Delta x}.$$

Subtract.

$$\therefore\ \Delta y=2\sqrt{x_0+\Delta x}-2\sqrt{x_0}.$$

Divide by Δx.

$$\therefore\ \frac{\Delta y}{\Delta x}=\frac{2\sqrt{x_0+\Delta x}-2\sqrt{x_0}}{\Delta x}.$$

$$\therefore \frac{\Delta y}{\Delta x} = \frac{2}{\sqrt{x_0 + \Delta x} + \sqrt{x_0}} \text{ (see Example 3, Art. 38).}$$

$$\therefore \tan \tau = \left.\frac{dy}{dx}\right|_{x=x_0} = \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right] = \frac{1}{\sqrt{x_0}}.$$

Therefore the slope of the tangent line to the curve $y = +2\sqrt{x}$ or the part of the parabola $y^2 = 4x$ above the x-axis, at any point (x_0, y_0) on the curve, is $\frac{1}{\sqrt{x_0}}$. In particular, if $x_0 = 1$, $\tan \tau = 1$, or $\tau = 45°$.

EXERCISES

1. In the curve $y = -2\sqrt{x}$, find the slope of the tangent line at any point (x_0, y_0) on the curve. *Ans.* $-\frac{1}{\sqrt{x_0}}$.

2. What is the angle which the tangent line to the parabola $y^2 = 4x$ makes with the x-axis at the origin? How does the angle change as the point (x_0, y_0) on the parabola is made to move farther out on the part of the curve above the x-axis? On the part below the x-axis?

3. Find the angle which the tangent line to the curve $y = x^3$ makes with the x-axis at the point $(-1, -1)$; at the point $(0, 0)$; at the point $(2, 8)$. *Ans.* $\tan^{-1} 3$, 0, $\tan^{-1} 12$.

Given that the equations of the tangent and normal lines to the curve $y = f(x)$ at a point (x_0, y_0) on the curve are

$$y - y_0 = \left.\frac{dy}{dx}\right|_{x=x_0} (x - x_0), \quad \text{equation of the tangent line,}$$

$$y - y_0 = -\frac{1}{\left.\frac{dy}{dx}\right|_{x=x_0}} (x - x_0), \quad \text{equation of the normal line,}$$

find:

4. The equations of the tangent and normal lines to the parabola $x^2 = 4y$, at the point $(2, 1)$; also at the point $(-2, 1)$.

Ans. $x - y = 1$.
$x + y = -1$.

5. The equations of the tangent and normal lines to the ellipse $4x^2 + y^2 = 8$, at the point $(1, 2)$; also at the point $(-1, 2)$.

Ans. $2x + y = 4.$
$2x - y = -4.$

41. Definitions. The mean or average speed of a moving body is the distance passed over by the body in a given time, divided by the time, the distance being expressed in units of length and the time in units of time.

For example, if a body passes over 60 feet in 2 seconds, its mean or average speed is 30 feet per second.

The actual speed of a moving body at a given instant is the limit which the mean speed for a period of time immediately succeeding the instant in question approaches as the length of the period of time is allowed to become indefinitely decreased.

The mean angular speed of a body rotating in a plane about a point is the angle through which the body has turned in a given time divided by the time, the angle being expressed in units of angle and the time in units of time.

For example, if a body rotates through an angle of 60° in 10 seconds, its mean or average angular speed is 6° per second.

The actual angular speed of a body, rotating in a plane about a point, at any instant, is the limit which the mean speed for a period of time immediately succeeding the instant in question approaches as the length of the period of time is allowed to become indefinitely decreased.

When there is danger of confusion between speed along a curve and angular speed, the former is called **linear speed.**

The speed of a moving body is said to be **uniform** when the actual speed at every instant is the same.

The speed of a moving body is said to be **variable** when the actual speed is continually changing.

Actual speed is usually spoken of simply as speed.

Another name for speed is **rate of motion,** or more simply rate.

42. **Units of speed.** The measure of mean linear speed is that of a certain distance divided by that of a certain time. The unit of mean linear speed is therefore the speed of a certain point which moves through a unit of distance in a unit of time. Expressed in English units it may be 1 foot per second, 1 mile per hour, etc., and in French units 1 centimeter per second, 1 kilometer per hour, etc. The measure of a mean angular speed is that of a certain angle divided by that of a certain time. The unit of angle usually employed is the radian. The unit of mean angular speed in terms of the radian is therefore 1 radian per unit of time.

Since the actual speed of a body at a point is expressed in terms of the limit of mean speed, its unit is that of the mean speed.

43. To find the speed of a body at any particular instant, we must find the mean speed of the body for a period of time immediately succeeding the instant in question, and then find the limit which this mean speed approaches as the period of time is allowed to become indefinitely decreased.

As an example in finding the speed of a moving body at any particular instant, consider the following problem:

A body falls freely from rest in a vacuum. Find its speed at the end of t_0 seconds.

By experiment, it has been found that the distance in feet passed over by a body falling freely from rest in a vacuum during a time t in seconds is $s = 16\,t^2$.

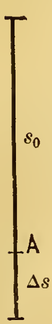


Fig. 13.

Then, in the equation $s = 16\,t^2$,

let $t = t_0$. $\therefore\ s_0 = 16\,t_0^2 =$ distance in feet the body has fallen in t_0 seconds.

Let $t = t_0 + \Delta t$. $\therefore\ s_0 + \Delta s = 16\,(t_0 + \Delta t)^2 =$ distance in feet the body has fallen in $t_0 + \Delta t$ seconds.

$\therefore\ \Delta s = 16\,(t_0 + \Delta t)^2 - 16\,t_0^2 =$ distance in feet the body has fallen in Δt seconds from point A. (Fig. 13.)

$\therefore \dfrac{\Delta s}{\Delta t} = 32\,t_0 + 16\,\Delta t =$ mean speed of the body over the distance Δs, expressed in feet per second.

$\therefore \left.\dfrac{ds}{dt}\right|_{t=t_0} = \underset{\Delta t \doteq 0}{\text{limit}}\left[\dfrac{\Delta s}{\Delta t}\right] = 32\,t_0 =$ actual speed of the body at the point A, or at the end of the time t_0, expressed in feet per second.

NOTE. The distance in feet passed over by a body falling freely from rest in a vacuum during a time t in seconds is $s = \frac{1}{2}gt^2$, where

$$g = 32.0894(1 + 0.0052375 \sin l)(1 - 0.0000000957\,e),^*$$

l being the latitude of the place and e the height in feet above the sea level. This gives g the value 32.2 approximately. In this problem it was chosen 32.

44. Definition. The equation by which the distance s passed over by a moving body in a time t is expressed as a function of t, is called the **law of motion** of the body.

Thus, in the example of the preceding article the distance s passed over by a body falling freely from rest in a vacuum during a time t is $s = \frac{1}{2}gt^2$. Then $s = \frac{1}{2}gt^2$ is the law of motion of this body.

In all cases the law of motion of the body must be determined before the speed of the body at any given instant can be found. This law is always derived with more or less mathematical manipulation from observed measurements. Thus, the law of the example of Art. 43 was determined by experiment.

45. As an example in which mathematical manipulation must be combined with observed measurements in order to determine the given law, consider the following problem:

One end of a ladder 20 feet long rests on the ground, 12 feet from the foundation of a building. The other rests against the side of the building. If the end on the ground is being carried away from the building on a line perpendicular

* See Maurer's *Technical Mechanics*, Part I, Art. 13.

to it at the uniform rate of 4 feet per second, find the law of motion of the other end.

Here, the observed measurements are the rate at which the end of the ladder on the ground is moving, the length of the ladder, and the distance of the foot of the ladder from the building when in its initial position. To determine the law from these measurements:

Let AB (Fig. 14) represent the side of the building, and BC the line perpendicular to it. Let t represent the number of seconds during which the ladder is moving. Then $12+4t$ is the distance of its foot from the building at the end of the time t. Let s represent the distance through which the top of the ladder has moved. If D is the point at which the ladder rested in its initial position, and F the point it has reached at the end of t seconds, then

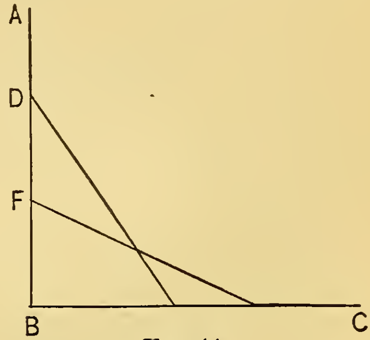


FIG. 14.

$$\begin{aligned} s &= DF \\ &= DB - FB \\ &= \sqrt{400-144} - \sqrt{400-(12+4t)^2} \\ &= 16 - 4\sqrt{16-6t-t^2}. \end{aligned}$$

Therefore $s = 16 - 4\sqrt{16-6t-t^2}$ is the law of motion of the top of the ladder.

46. If the law of motion of a body can be formulated so as to express the distance s passed over by the body as a function of the time t, then the speed of the body at the time t, namely $\left.\frac{ds}{dt}\right|_{t=t_0}$, can be determined as in the example of Art. 43.

As a further illustration of finding the speed of a body at a given instant when the law of motion is known, find the

speed at which the top of the ladder in the problem of the preceding article is moving at the end of (*a*) 1 second; (*b*) $1\frac{1}{2}$ seconds.

To find these speeds, find $\left.\frac{ds}{dt}\right|_{t=t_0}$, and substitute 1 and $1\frac{1}{2}$ in turn in the result.

Let $t = t_0$. $\therefore\ s_0 = 16 - 4\sqrt{16 - 6t_0 - t_0^2}$.

Let $t = t_0 + \Delta t$. $\therefore\ s_0 + \Delta s = 16 - 4\sqrt{16 - 6(t_0 + \Delta t) - (t_0 + \Delta t)^2}$.

$$\therefore\ \frac{\Delta s}{\Delta t} = -4\frac{\sqrt{16 - 6(t_0 + \Delta t) - (t_0 + \Delta t)^2} - \sqrt{16 - 6t_0 - t_0^2}}{\Delta t}.$$

Rationalize the numerator and pass to limits.

$$\therefore\ \left.\frac{ds}{dt}\right|_{t=t_0} = 4\frac{t_0 + 3}{\sqrt{16 - 6t_0 - t_0^2}}.$$

$$\therefore\ \left.\frac{ds}{dt}\right|_{t=1} = 4\frac{4}{\sqrt{9}} = \frac{16}{3} \text{ feet per second, and}$$

$$\left.\frac{ds}{dt}\right|_{t=1\frac{1}{2}} = 4\frac{4\frac{1}{2}}{\sqrt{\frac{19}{4}}} = \tfrac{36}{19}\sqrt{19} \text{ feet per second.}$$

EXERCISES

1. A man 6 feet high walks directly away from a lamp-post 10 feet high at the uniform rate of 4 miles per hour. How fast does the end of his shadow move? Does it move uniformly? *Ans.* 10 mi. per hr. Yes.

2. In Exercise 1, how fast is his shadow lengthening? Does it lengthen uniformly? *Ans.* 6 mi. per hr. Yes.

3. A man is walking up a plane inclined 45° to the ground at the rate of 2 miles per hour. At what rate is his projection on the ground moving? *Ans.* $\sqrt{2}$ mi. per hr.

4. At what rate must a man walk up a plane inclined 60° to the ground in order that his projection on a plane perpendicular to the ground may move at the rate of 1 mile per hour? *Ans.* $\frac{2}{\sqrt{3}}$ mi. per hr.

5. The radius of a spherical soap bubble is increasing uniformly at the rate of $\frac{1}{10}$ inch per second. Find the rate at which the volume is increasing when the diameter is $\frac{1}{20}$ inch. *Ans.* $\frac{\pi}{4000}$ cu. in. per sec.

If a body with an initial speed v_0 moves down a smooth plane inclined $\alpha°$ to the horizontal, the law of motion of the body is

$$s = \tfrac{1}{2} g t^2 \sin \alpha + v_0 t.$$

Assuming this law, solve the following problems. $(g = 32.)$

6. If a body starts from rest and moves down a smooth plane inclined 60° to the horizontal, find its speed 1 second after it starts. *Ans.* $16\sqrt{3}$ ft. per sec.

7. If a body with an initial speed of 10 feet per second moves down a smooth plane inclined 30° to the horizontal, find its speed when it has gone 10 yards, and its mean speed for this distance. *Ans.* 32.56 ft. per sec.; 21.28 ft. per sec.

47. So far we have considered the speed of a body when the law of motion can be formulated so as to express the distance passed over as a function of the time. We shall now consider a slightly more general problem, one in which the ratio of the speeds of two bodies can be determined when the law of motion can be formulated so as to express the distance passed over by one as a function of that passed over by the other in the same time.

48. As an example, consider the following problem:

Suppose that two bodies are moving in such a manner that in a time t one passes over a distance y and the other over a distance x, where $y = x^2 + 3x + 2$. To find the ratio of the speeds of the two bodies at the end of a time t_0.

At the end of a time t_0, one has passed over a distance x_0 and the other over a distance y_0, where

$$y_0 = x_0^2 + 3x_0 + 2.$$

At the end of a time $t_0 + \Delta t$, one has passed over a distance $x_0 + \Delta x$ and the other over a distance $y_0 + \Delta y$, where

$$y_0 + \Delta y = (x_0 + \Delta x)^2 + 3(x_0 + \Delta x) + 2.$$

$$\therefore\ \Delta y = (x_0 + \Delta x)^2 + 3(x_0 + \Delta x) + 2 - (x_0^2 + 3x_0 + 2)$$

$$= (2x_0 + 3)\Delta x + \overline{\Delta x}^2.$$

$$\therefore\ \frac{\Delta y}{\Delta t} = \frac{(2x_0 + 3)\Delta x + \overline{\Delta x}^2}{\Delta t}$$

$$= \frac{(2x_0 + 3)\Delta x + \Delta x^2}{\Delta x} \cdot \frac{\Delta x}{\Delta t},$$

by dividing and multiplying by Δx.

$$\therefore\ \underset{\Delta t \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta t}\right] = \underset{\Delta t \doteq 0}{\text{limit}} \left[\frac{(2x_0 + 3)\Delta x + \overline{\Delta x}^2}{\Delta x}\right] \cdot \underset{\Delta t \doteq 0}{\text{limit}} \left[\frac{\Delta x}{\Delta t}\right].$$

Now as $\Delta t \doteq 0$, $\Delta x \doteq 0$.

$$\therefore\ \underset{\Delta t \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta t}\right] = \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{(2x_0 + 3)\Delta x + \overline{\Delta x}^2}{\Delta x}\right] \cdot \underset{\Delta t \doteq 0}{\text{limit}} \left[\frac{\Delta x}{\Delta t}\right].$$

$$\therefore\ \left.\frac{dy}{dt}\right|_{t=t_0} = (2x_0 + 3) \left.\frac{dx}{dt}\right|_{t=t_0}.$$

Now $\left.\frac{dx}{dt}\right|_{t=t_0}$ and $\left.\frac{dy}{dt}\right|_{t=t_0}$ are the speeds at the end of a time t_0 of the bodies that move over distances x and y respectively in a time t. Therefore the ratio of the speeds of the two bodies is such that

$$\left.\frac{dy}{dt}\right|_{t=t_0} \div \left.\frac{dx}{dt}\right|_{t=t_0} = 2x_0 + 3.$$

It will be noticed that $2x_0 + 3$ is $\left.\frac{dy}{dx}\right|_{x=x_0}$. Therefore, in this example,

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \frac{\left.\frac{dy}{dt}\right|_{t=t_0}}{\left.\frac{dx}{dt}\right|_{t=t_0}}.$$

49. It can be readily proved in general that if the distances passed over by two bodies in a time t are x and y respectively, and y can be expressed as a function of x, then

$$\left.\frac{dy}{dx}\right|_{x=x_0}=\frac{\left.\frac{dy}{dt}\right|_{t=t_0}}{\left.\frac{dx}{dt}\right|_{t=t_0}},$$

where x_0 is the value of x when $t=t_0$.

Proof. For any value t_0 of t,

$$\frac{\Delta y}{\Delta t}=\frac{\Delta y}{\Delta x}\cdot\frac{\Delta x}{\Delta t}, \text{ identically.}$$

$$\therefore \operatorname*{limit}_{\Delta t\doteq 0}\left[\frac{\Delta y}{\Delta t}\right]=\operatorname*{limit}_{\Delta t\doteq 0}\left[\frac{\Delta y}{\Delta x}\right]\cdot\operatorname*{limit}_{\Delta t\doteq 0}\left[\frac{\Delta x}{\Delta t}\right].$$

Now as $\Delta t\doteq 0$, $\Delta x\doteq 0$.

$$\therefore \operatorname*{limit}_{\Delta t\doteq 0}\left[\frac{\Delta y}{\Delta t}\right]=\operatorname*{limit}_{\Delta x\doteq 0}\left[\frac{\Delta y}{\Delta x}\right]\cdot\operatorname*{limit}_{\Delta t\doteq 0}\left[\frac{\Delta x}{\Delta t}\right].$$

$$\therefore \left.\frac{dy}{dt}\right|_{t=t_0}=\left.\frac{dy}{dx}\right|_{x=x_0}\cdot\left.\frac{dx}{dt}\right|_{t=t_0}.$$

$$\therefore \left.\frac{dy}{dx}\right|_{x=x_0}=\frac{\left.\frac{dy}{dt}\right|_{t=t_0}}{\left.\frac{dx}{dt}\right|_{t=t_0}}.$$

50. In the curve $y=f(x)$, x is the abscissa and y the ordinate of any point (x, y) on the curve. If we look on the curve as being generated by a moving point whose coördinates are (x, y), then $\left.\frac{dx}{dt}\right|_{t=t_0}$ and $\left.\frac{dy}{dt}\right|_{t=t_0}$ are the rates at which the point is moving parallel to the x and y axes respectively at the end of a time t_0. In other words, $\left.\frac{dx}{dt}\right|_{t=t_0}$ and $\left.\frac{dy}{dt}\right|_{t=t_0}$ are the rates at which the abscissa and ordinate respectively

are changing in length at the end of the time t_0. Therefore $\left.\frac{dy}{dx}\right|_{x=x_0}$ is the ratio of the rate of change in length of the ordinate to that of the abscissa of the curve at a point (x_0, y_0). As seen in Art. 39, it is also the slope of the tangent line to the curve at this point.

EXERCISES

1. If $y=3x^2-2x+4$, find the ratio of the rate of change of y to that of x when $x=2$. *Ans.* 10.

2. In $y=x^2+3x-2$, what is the value of x at the point where y increases 6 times as fast as x? *Ans.* $x=\frac{3}{2}$.

3. For what values of x do $y=x^3-3x^2+2x-2$ and $y=x^2-6$ change at the same rate? *Ans.* $x=\frac{4\pm\sqrt{10}}{3}$.

4. Find the coördinates of the point at which $y=3x^2+2x-2$ changes at the same rate as the slope of the tangent line to the curve that represents this equation, at the point. *Ans.* $(\frac{2}{3}, \frac{2}{3})$.

Water flows from a vessel, in the form of a right circular cylinder of radius 1 foot, into one of the form of an inverted circular cone of semivertical angle 30°:

5. If the level of the water in the cylinder falls uniformly at the rate of 2 inches per minute, at what rate is the water flowing? *Ans.* 288π cu. in. per min.

6. With the above rate of flow, at what rate will the level of the water in the cone be rising when the depth is 6 inches? When 1 foot? When 2 feet? *Ans.* 24, 6, 1.5 in. per min.

7. At what depth of water in the cone will the level be rising just as fast as it is falling in the cylinder? At what depth, three times as fast? *Ans.* 20.78 in.; 12 in.

CHAPTER IV

GENERAL FORMULAS FOR DIFFERENTIATION

51. Definition. The process of finding the derivative of a function is called differentiation.

In all cases differentiation may be performed by following the steps mentioned in Art. 38. It is, however, inconvenient to be compelled to resort to this method in each particular problem, so we shall derive rules or formulas, called **formulas for differentiation**, by the aid of which, labor may be avoided in finding the derivative of a function. In the next article seven such formulas are derived. They are numbered from I to VII for convenience of reference. As will be seen, they apply to all the classes of functions given in Art. 7. Hence the name "General Formulas."

52. GENERAL FORMULAS FOR DIFFERENTIATION

I. $$\frac{dx}{dx} = 1.$$

II. $$\frac{dc}{dx} = 0.$$

III. $$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

IV. $$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

V. $$\frac{d(cu)}{dx} = c\frac{du}{dx}.$$

VI. $$\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

VII. $$\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}.$$

In these formulas, u and v are functions of the independent variable x; c is any constant; and n is any rational constant.

NOTE. Formula VII is true for n any constant, but we shall consider it only in the cases where n is a rational constant.

53. Derivation of formulas.

Proof of I.

Let $\quad y = x.$

Let $x = x_0$. $\quad \therefore\ y_0 = x_0.$

Let $x = x_0 + \Delta x$. $\quad \therefore\ y_0 + \Delta y = x_0 + \Delta x.$

Subtract. $\quad \therefore\ \Delta y = \Delta x.$

Divide by Δx. $\quad \therefore\ \dfrac{\Delta y}{\Delta x} = 1.$

Pass to limits. $\quad \therefore\ \left.\dfrac{dy}{dx}\right|_{x=x_0} = \underset{\Delta x \doteq 0}{\text{limit}}\left[\dfrac{\Delta y}{\Delta x}\right] = 1.$

$$\therefore\ \frac{dy}{dx} = 1, \text{ or since } y = x,\ \frac{dx}{dx} = 1.$$

Proof of II.

Let $\quad y = c.$

Let $x = x_0$. $\quad \therefore\ y_0 = c.$

Let $x = x_0 + \Delta x$. $\quad \therefore\ y_0 + \Delta y = c.$

Subtract. $\quad \therefore\ \Delta y = 0.$

Divide by Δx. $\quad \therefore\ \dfrac{\Delta y}{\Delta x} = 0.$

Pass to limits. $\quad \therefore\ \left.\dfrac{dy}{dx}\right|_{x=x_0} = \underset{\Delta x \doteq 0}{\text{limit}}\left[\dfrac{\Delta y}{\Delta x}\right] = 0.$

$$\therefore\ \frac{dy}{dx} = 0, \text{ or since } y = c,\ \frac{dc}{dx} = 0.$$

Proof of III.

Let $$y = u + v.$$

Let $x = x_0$. $$\therefore\ y_0 = u_0 + v_0.$$

Let $x = x_0 + \Delta x$. $$\therefore\ y_0 + \Delta y = u_0 + \Delta u + v_0 + \Delta v.$$

Subtract. $$\therefore\ \Delta y = \Delta u + \Delta v.$$

Divide by Δx. $$\therefore\ \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

Pass to limits. $$\therefore\ \frac{dy}{dx}\bigg|_{x=x_0} = \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}\right]$$

$$= \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta u}{\Delta x}\right] + \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta v}{\Delta x}\right]$$

$$= \frac{du}{dx}\bigg|_{x=x_0} + \frac{dv}{dx}\bigg|_{x=x_0}.$$

$$\therefore\ \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}, \text{ or since } y = u + v,\quad \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

A similar proof will apply to any number of functions, so that

$$\frac{d(u + v + w + \cdots)}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \cdots.$$

Proof of IV.

Let $$y = uv.$$

Let $x = x_0$. $$\therefore\ y = u_0 v_0.$$

Let $x = x_0 + \Delta x$. $$\therefore\ y_0 + \Delta y = (u_0 + \Delta u)(v_0 + \Delta v).$$

Subtract. $$\therefore\ \Delta y = (u_0 + \Delta u)(v_0 + \Delta v) - u_0 v_0$$

$$= u_0 \Delta v + v_0 \Delta u + \Delta u \Delta v.$$

Divide by Δx. $$\therefore\ \frac{\Delta y}{\Delta x} = u_0 \frac{\Delta v}{\Delta x} + v_0 \frac{\Delta u}{\Delta x} + \Delta u \frac{\Delta v}{\Delta x}.$$

Pass to limits. $$\therefore \left.\frac{dy}{dx}\right|_{x=x_0} = \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta x}\right]$$
$$= \underset{\Delta x \doteq 0}{\text{limit}}\left[u_0\frac{\Delta v}{\Delta x} + v_0\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}\right]$$
$$= \underset{\Delta x \doteq 0}{\text{limit}}\left[u_0\frac{\Delta v}{\Delta x}\right] + \underset{\Delta x \doteq 0}{\text{limit}}\left[v_0\frac{\Delta u}{\Delta x}\right]$$
$$+ \underset{\Delta x \doteq 0}{\text{limit}}\left[\Delta u\frac{\Delta v}{\Delta x}\right].$$

Now $$\underset{\Delta x \doteq 0}{\text{limit}}\left[u_0\frac{\Delta v}{\Delta x}\right] = u_0\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta v}{\Delta x}\right] = u_0\left.\frac{dv}{dx}\right|_{x=x_0},$$
$$\underset{\Delta x \doteq 0}{\text{limit}}\left[v_0\frac{\Delta u}{\Delta x}\right] = v_0\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta u}{\Delta x}\right] = v_0\left.\frac{du}{dx}\right|_{x=x_0},$$
and $$\underset{\Delta x \doteq 0}{\text{limit}}\left[\Delta u\cdot\frac{\Delta v}{\Delta u}\right] = \underset{\Delta x \doteq 0}{\text{limit}}\left[\Delta u\right]\cdot\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta v}{\Delta x}\right] = 0\cdot\left.\frac{dv}{dx}\right|_{x=x_0} = 0.$$
$$\therefore \left.\frac{d(uv)}{dx}\right|_{x=x_0} = u_0\left.\frac{dv}{dx}\right|_{x=x_0} + v_0\left.\frac{du}{dx}\right|_{x=x_0}.$$
$$\therefore \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Proof of V.

Let $$y = cu.$$

Let $x = x_0$. $$\therefore y_0 = cu_0.$$

Let $x = x_0 + \Delta x$. $$\therefore y_0 + \Delta y = c(u_0 + \Delta u).$$

Subtract. $$\therefore \Delta y = c\Delta u.$$

Divide by Δx. $$\therefore \frac{\Delta y}{\Delta x} = c\frac{\Delta u}{\Delta x}.$$

Pass to limits. $$\therefore \left.\frac{dy}{dx}\right|_{x=x_0} = \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta x}\right] = \underset{\Delta x \doteq 0}{\text{limit}}\left[c\frac{\Delta u}{\Delta x}\right]$$
$$= c\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta u}{\Delta x}\right]$$
$$= c\left.\frac{du}{dx}\right|_{x=x_0}.$$
$$\therefore \frac{d(cu)}{dx} = c\frac{du}{dx}.$$

Proof of VI.

Let $$y = \frac{u}{v}.$$

Let $x = x_0$. $$\therefore\ y_0 = \frac{u_0}{v_0}.$$

Let $x = x_0 + \Delta x$. $$\therefore\ y_0 + \Delta y = \frac{u_0 + \Delta u}{v_0 + \Delta v}.$$

Subtract. $$\therefore\ \Delta y = \frac{u_0 + \Delta u}{v_0 + \Delta v} - \frac{u_0}{v_0}$$

$$= \frac{v_0 \Delta u - u_0 \Delta v}{v_0(v_0 + \Delta v)}.$$

Divide by Δx. $$\therefore\ \frac{\Delta y}{\Delta x} = \frac{v_0 \dfrac{\Delta u}{\Delta x} - u_0 \dfrac{\Delta v}{\Delta x}}{v_0(v_0 + \Delta v)}.$$

Pass to limits. $$\therefore\ \left.\frac{dy}{dx}\right|_{x=x_0} = \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right] = \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{v_0 \dfrac{\Delta u}{\Delta x} - u_0 \dfrac{\Delta v}{\Delta x}}{v_0(v_0 + \Delta v)}\right]$$

$$= \frac{v_0 \underset{\Delta x \doteq 0}{\text{limit}} \left[\dfrac{\Delta u}{\Delta x}\right] - u_0 \underset{\Delta x \doteq 0}{\text{limit}} \left[\dfrac{\Delta v}{\Delta x}\right]}{v_0 \underset{\Delta x \doteq 0}{\text{limit}} \left[v_0 + \Delta v\right]}$$

$$= \frac{v_0 \left.\dfrac{du}{dx}\right|_{x=x_0} - u_0 \left.\dfrac{dv}{dx}\right|_{x=x_0}}{{v_0}^2}.$$

$$\therefore\ \frac{d\left(\dfrac{u}{v}\right)}{dx} = \frac{v \dfrac{du}{dx} - u \dfrac{dv}{dx}}{v^2}.$$

Proof of VII.

Let $$y = u^n.$$

At first suppose that n is a positive integer.

Let $x = x_0$. $$\therefore\ y_0 = {u_0}^n.$$

Let $x = x_0 + \Delta x$. $$\therefore\ y_0 + \Delta y = (u_0 + \Delta u)^n.$$

Subtract. $$\therefore\ \Delta y = (u_0 + \Delta u)^n - {u_0}^n.$$

Now $(u_0+\Delta u)^n - u_0^n$ can be factored like $a^n - u_0^n$. Let $u_0+\Delta u = a$.

The factors are $a-u_0$ and $(a^{n-1}+a^{n-2}u_0+a^{n-3}u_0^2+\cdots+au_0^{n-2}+u_0^{n-1})$.

Substitute $u_0+\Delta u$ for a in these factors.

$$\therefore\ (u_0+\Delta u)^n - u_0^n = \Delta u\,\{(u_0+\Delta u)^{n-1}+u_0(u_0+\Delta u)^{n-2}$$
$$+u_0^2(u_0+\Delta u)^{n-3}+\cdots+u_0^{n-2}(u_0+\Delta u)+u_0^{n-1}\}.$$

Divide by Δx.

$$\therefore\ \frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x}\{(u_0+\Delta u)^{n-1}+u_0(u_0+\Delta u)^{n-2}$$
$$+u_0^2(u_0+\Delta u)^{n-3}+\cdots+u_0^{n-1}\}.$$

Pass to limits.

$$\therefore\ \left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \doteq 0}\left[\frac{\Delta y}{\Delta x}\right]$$

$$= \lim_{\Delta x \doteq 0}\left[\frac{\Delta u}{\Delta x}\{(u_0+\Delta u)^{n-1}+u_0(u_0+\Delta u)^{n-2}+u_0^2(u_0+\Delta u)^{n-3}+\cdots+u_0^{n-1}\}\right].$$

As Δx approaches zero, Δu approaches zero, and

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{du}{dx}\right|_{x=x_0}\{u_0^{n-1}+u_0^{n-1}+u_0^{n-1}+\cdots+u_0^{n-1}\},$$

there being n terms in the parenthesis,

$$= nu_0^{n-1}\left.\frac{du}{dx}\right|_{x=x_0}.$$

$$\therefore\ \frac{du^n}{dx} = nu^{n-1}\frac{du}{dx},$$ when n is a positive integer.

Next, suppose that n is a positive fraction $=\frac{p}{q}$, where p and q are positive integers.

Then $y = u^{\frac{p}{q}}$.

Raise both sides of the equation to the qth power.

$$\therefore\ y^q = u^p.$$

$$\therefore\ \frac{dy^q}{dx} = \frac{du^p}{dx}.$$

Now u^p is of the form u^n where n is a positive integer, because p is a positive integer.

$$\therefore \frac{du^p}{dx} = pu^{p-1}\frac{du}{dx}.$$

Also, y^q is of the form u^n where n is a positive integer, because q is a positive integer.

$$\therefore \frac{dy^q}{dx} = qy^{q-1}\frac{dy}{dx}.$$

$$\therefore qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx}.$$

$$\therefore \frac{dy}{dx} = \frac{p}{q} \cdot \frac{u^{p-1}}{y^{q-1}}\frac{du}{dx}$$

$$= \frac{p}{q}\frac{u^{p-1}}{\left(u^{\frac{p}{q}}\right)^{q-1}}\frac{du}{dx}$$

$$= \frac{p}{q}\frac{u^{p-1}}{u^p \cdot u^{-\frac{p}{q}}}\frac{du}{dx}$$

$$= \frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx}$$

$$= nu^{n-1}\frac{du}{dx}.$$

$$\therefore \frac{du^n}{dx} = nu^{n-1}\frac{du}{dx} \text{ if } n \text{ is a positive fraction.}$$

Next, suppose that n is negative, $= -m$, either integral or fractional.

Then $y = u^n = u^{-m}$ where m is positive,

$$= \frac{1}{u^m}.$$

$$\therefore \frac{dy}{dx} = \frac{u^m \frac{d \cdot 1}{dx} - 1 \cdot \frac{du^m}{dx}}{u^{2m}}, \qquad \text{by Formula VI,}$$

$$= -\frac{mu^{m-1}}{u^{2m}} \frac{du}{dx}, \quad \text{by Formulas II and VII,}$$

$$= -mu^{-m-1}\frac{du}{dx}$$

$$= nu^{n-1}\frac{du}{dx}, \quad \text{since } n = -m.$$

Therefore $\frac{du^n}{dx} = nu^{n-1}\frac{du}{dx}$ for n an integer or fraction, positive or negative.

54. The way in which these formulas may be employed in finding the derivative of a function is illustrated in the following examples:

EXAMPLE 1. In the equation $y = x^4$, find $\frac{dy}{dx}$.

Let $u = x$ and Formula VII applies directly.

$$\therefore \frac{dy}{dx} = 4x^3\frac{dx}{dx}$$

$$= 4x^3, \text{ since } \frac{dx}{dx} = 1 \text{ by Formula I.}$$

$$\therefore \frac{dy}{dx} = 4x^3.$$

EXAMPLE 2. In the equation $y = 4x^3 - \frac{1}{2}x + 1$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{d(4x^3 - \frac{1}{2}x + 1)}{dx}$$

$$= \frac{d\,4x^3}{dx} + \frac{d(-\frac{1}{2}x)}{dx} + \frac{d \cdot 1}{dx}, \text{ by Formula III,}$$

$$= 4\frac{dx^3}{dx} - \frac{1}{2}\frac{dx}{dx} + \frac{d \cdot 1}{dx}, \text{ by Formula V,}$$

$$= 12x^2 - \frac{1}{2}, \text{ by Formulas VII, I and II.}$$

EXAMPLE 3. In the equation $y = 3\sqrt{x} + \dfrac{1}{2\sqrt{x}} + 4\sqrt{x^3}$, find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = 3\frac{d\sqrt{x}}{dx} + \frac{1}{2}\frac{d\left(\frac{1}{\sqrt{x}}\right)}{dx} + 4\frac{d\sqrt{x^3}}{dx}$$

$$= 3\frac{dx^{\frac{1}{2}}}{dx} + \frac{1}{2}\frac{dx^{-\frac{1}{2}}}{dx} + 4\frac{dx^{\frac{3}{2}}}{dx}$$

$$= \tfrac{3}{2}x^{-\frac{1}{2}} - \tfrac{1}{4}x^{-\frac{3}{2}} + 6x^{\frac{1}{2}}, \text{ by Formula VII,}$$

$$= \frac{3}{2\sqrt{x}} - \frac{1}{4\sqrt{x^3}} + 6\sqrt{x}.$$

EXAMPLE 4. In the equation $y = (x-1)\sqrt{x^2+1}$, find $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = (x-1)\frac{d\sqrt{x^2+1}}{dx} + \sqrt{x^2+1}\,\frac{d(x-1)}{dx}, \text{ by Formula IV.}$$

Let $x^2+1 = u$. $\therefore\ \dfrac{d\sqrt{x^2+1}}{dx} = \dfrac{d\sqrt{u}}{dx} = \dfrac{1}{2}u^{-\frac{1}{2}}\dfrac{du}{dx}$

$$= \frac{1}{2}(x^2+1)^{-\frac{1}{2}}\frac{d(x^2+1)}{dx}$$

$$= \frac{x}{\sqrt{x^2+1}}.$$

$$\therefore\ \frac{dy}{dx} = \frac{(x-1)x}{\sqrt{x^2+1}} + \sqrt{x^2+1}$$

$$= \frac{2x^2 - x + 1}{\sqrt{x^2+1}}.$$

EXERCISES

Find $\dfrac{dy}{dx}$ in each of the following cases:

1. $y = \dfrac{2 + 3cx + 4bx^2}{4x}$. *Ans.* $-\dfrac{1}{2x^2} + b$.

2. $y = \dfrac{x^{\frac{1}{6}} + x^{\frac{1}{2}} - x}{x^{\frac{1}{3}}}$. *Ans.* $-\dfrac{1}{6x^{\frac{7}{6}}} + \dfrac{1}{6x^{\frac{5}{6}}} - \dfrac{2}{3x^{\frac{1}{3}}}$.

3. $y=\dfrac{x^{\frac{7}{2}}+x^{\frac{3}{4}}-x^{\frac{1}{2}}+1}{x}$. *Ans.* $\dfrac{5}{2}x^{\frac{3}{2}}-\dfrac{1}{4x^{\frac{5}{4}}}+\dfrac{1}{2x^{\frac{3}{2}}}-\dfrac{1}{x^2}$.

4. $y=(1+2x+3x^2+4x^3)(1-x)^2$. *Ans.* $-20x^3(1-x)$.

5. $y=(2x^3-1)(1+x^3)^2$. *Ans.* $18x^5(x^3+1)$.

6. $y=\dfrac{cx}{\sqrt{x-1}}$. *Ans.* $\dfrac{c(x-2)}{2(x-1)\sqrt{x-1}}$.

7. $y=\dfrac{cx}{\sqrt{x+1}}$. *Ans.* $\dfrac{c(x+2)}{2(x+1)\sqrt{x+1}}$.

8. $y=\sqrt{\dfrac{x-1}{x+1}}$. *Ans.* $\dfrac{1}{(x+1)\sqrt{x^2-1}}$.

9. $y=\sqrt{\dfrac{x+1}{x-1}}$. *Ans.* $\dfrac{-1}{(x-1)\sqrt{x^2-1}}$.

10. $y=\dfrac{\sqrt{a+x}}{\sqrt{a}+\sqrt{x}}$. *Ans.* $\dfrac{\sqrt{a}(x-a)}{2\sqrt{x}\sqrt{a+x}(\sqrt{a}+\sqrt{x})^2}$.

11. $y=\dfrac{\sqrt{a}-\sqrt{x}}{\sqrt{a+x}}$. *Ans.* $\dfrac{-\sqrt{a}(\sqrt{a}+\sqrt{x})}{2\sqrt{x}(a+x)^{\frac{3}{2}}}$.

12. $y=\sqrt{\dfrac{1-x}{(1+x)^3}}$. *Ans.* $\dfrac{x-2}{\sqrt{(1-x)(1+x)^5}}$.

13. $y=\sqrt{1-x}\,\sqrt[3]{1+x}$. *Ans.* $-\dfrac{1+5x}{6\sqrt[6]{(1-x)^3(1+x)^4}}$.

14. $y=(x^2-1)\sqrt{x^3+1}$. *Ans.* $\dfrac{7x^4-3x^2+4x}{2\sqrt{x^3+1}}$.

15. $y=\dfrac{x}{\sqrt{1-x^2}}$. *Ans.* $\dfrac{1}{(1-x^2)\sqrt{1-x^2}}$.

16. $y=\dfrac{1-x}{\sqrt{1+x^2}}$. *Ans.* $-\dfrac{1+x}{(1+x^2)\sqrt{1+x^2}}$.

17. $y=\dfrac{\sqrt{x}}{1+x^2}$. *Ans.* $\dfrac{1-3x^2}{2\sqrt{x}(1+x^2)^2}$.

18. $y = \dfrac{1}{x - \sqrt{1 + x^2}}$. *Ans.* $\dfrac{1}{\sqrt{1 + x^2}\,(x - \sqrt{1 + x^2})}$.

19. $y = \dfrac{2x^2 + 1}{x(x^2 + 3)^{\frac{2}{3}}}$. *Ans.* $\dfrac{-2x^4 + 11x^2 - 9}{3x^2(x^2 + 3)^{\frac{5}{3}}}$.

20. $y = \dfrac{x\sqrt{(x^2 + 3)^3}}{x - 1}$. *Ans.* $\dfrac{\sqrt{x^2 + 3}\,(3x^3 - 4x^2 - 3)}{(x - 1)^2}$.

In the two following examples, find $\dfrac{dy}{dz}$, $\dfrac{dz}{dx}$, and then $\dfrac{dy}{dx}$. (See Art. 49.)

21. $y = z^3 - 3z$, $z = x^2 - x$.

Ans. $\dfrac{dy}{dz} = 3(z^2 - 1)$; $\dfrac{dz}{dx} = 2x - 1$; $\dfrac{dy}{dx} = 3(x^4 - 2x^3 + x^2 - 1)(2x - 1)$.

22. $y = 3z^2 - z + 1$, $z = x^2 - 1$.

Ans. $\dfrac{dy}{dz} = 6z - 1$; $\dfrac{dz}{dx} = 2x$; $\dfrac{dy}{dx} = 2(6x^2 - 7)x$.

23. In the curves $y = x^3 - 12x^2 + 4x$ and $y = x^3 - 8x^2 - 8$, given that the abscissas are changing at the same rate, what are the coördinates of the points at which the ordinates are changing at the same rate? *Ans.* $(\frac{1}{2}, -\frac{7}{8})$; $(\frac{1}{2}, -\frac{79}{8})$.

24. In the curves of Exercise 23, what is the ratio of the rate of change of the ordinate of the first to that of the second at the points where the curves cut each other?

Ans. $1\frac{3}{5}$ at $(2, -32)$; $\frac{31}{19}$ at $(-1, -17)$.

25. At what angles does the parabola $x^2 = 4ay$ intersect the witch $y = \dfrac{8a^3}{x^2 + 4a^2}$? *Ans.* $\tan^{-1} \pm 3$.

26. Find the points on the curve $y = (x + 1)(x - 2)(x - 3)$ at which the tangent line to the curve is parallel to the x-axis. Makes 45° with the x-axis.

Ans. $\left(\dfrac{4 + \sqrt{13}}{3}, \dfrac{70 - 26\sqrt{13}}{27}\right)$; $\left(\dfrac{4 - \sqrt{13}}{3}, \dfrac{70 + 26\sqrt{13}}{27}\right)$;

$(0, 6)$; $(\frac{8}{3}, -\frac{22}{27})$.

27. If $(x-a)$ occurs n times as a factor in a function $f(x)$, show that it occurs $(n-1)$ times in $\frac{df(x)}{dx}\cdot$ Hence devise a method for determining whether a double root occurs in an algebraic equation.

28. One end of a rope is attached to a boat at the surface of the water while the other end passes over a pulley 10 feet above the surface. If the rope is being shortened at the rate of 40 yards per minute, how fast is the boat moving through the water when it is 5 feet from the foot of the perpendicular from the pulley to the water? *Ans.* $2\sqrt{5}$ ft. per sec.

29. A man is walking at the rate of 4 miles per hour on a straight road from which a given point is distant 100 feet. At what rate is he approaching the point when he is 200 feet distant from the foot of the perpendicular from the point to the road? *Ans.* 5.2 ft. per sec.

30. One ship is 41 miles due north of another. The first is sailing south at the rate of 8 miles per hour, and the second is sailing east at the rate of 10 miles per hour. How rapidly are they approaching each other? How long will they continue to do so? *Ans.* 8 mi. per hr. For 2 hr.

31. Two railroad tracks intersect at a station at an angle of 60°. One train leaves the station on one track and travels at the rate of 30 miles per hour, and two minutes later another leaves on the other track and travels at the rate of 60 miles per hour. At what rate are the trains separating 10 minutes after the first started? *Ans.* 51.4 mi. per hr.

32. A reservoir in the form of the frustum of an inverted regular tetrahedron, whose lower edges are each 100 feet, and whose sides are inclined at an angle of 45° with the horizontal, is used to supply a town with water. If the depth of the water at any instant is 10 feet, and is falling at the rate of 3 feet per day, at what rate is the town being supplied?

Ans. 23548.8 cu. ft. per day.

CHAPTER V

SUCCESSIVE DIFFERENTIATION

55. The limit of the ratio of the increment of a function of one variable to the increment of the variable, as the increment of the variable approaches zero, is the derivative of the function with respect to the variable. This derivative is called the **first derivative** of the function with respect to the variable. Now this derivative is usually itself a function of the variable and can be differentiated with respect to it. The derivative of the first derivative is called the **second derivative** of the function with respect to the variable. The second derivative is usually also a function of the variable and can be differentiated with respect to it. The derivative of the second derivative is called the **third derivative** of the function with respect to the variable. And so on. In general, the derivative with respect to the variable of the $(n-1)$th derivative of a function with respect to that variable is called the n**th derivative** of the function with respect to that variable.

For example, in the equation $y = x^4$,

$\frac{dy}{dx} = 4x^3$, the first derivative of y with respect to x;

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 12x^2$, the second derivative of y with respect to x;

$\frac{d}{dx}\left(\frac{dy}{dx}\left(\frac{dy}{dx}\right)\right) = 24x$, the third derivative of y with respect to x;

$\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{dy}{dx}\right)\right)\right) = 24$, the fourth derivative of y with respect to x.

Since 24 is a constant, the fifth and all succeeding derivatives are zero.

56. Notation. In the equation $y=f(x)$,

$\dfrac{dy}{dx}$ is denoted sometimes by $f'(x)$;

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$ is denoted by $\dfrac{d^2y}{dx^2}$ or sometimes by $f''(x)$;

$\dfrac{d}{dx}\left(\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)\right)$ is denoted by $\dfrac{d^3y}{dx^3}$ or sometimes by $f'''(x)$.

The nth derivative of y with respect to x is denoted by $\dfrac{d^ny}{dx^n}$ or sometimes by $f^{(n)}(x)$.

EXERCISES

1. In the equation $y=x^5$, find $\dfrac{d^3y}{dx^3}$. *Ans.* $60\,x^2$.

2. In the equation $y=\dfrac{a}{x}$, find $\dfrac{d^2y}{dx^2}$. *Ans.* $\dfrac{2\,a}{x^3}$.

3. In the equation $y=2\,x^3-3\,x^2-12\,x+1$, find the values of x that make $f'(x)$ zero and $f''(x)$ positive. How many such values are there? *Ans.* 2. One value.

4. In the equation $y=x^3-6\,x^2+x-1$, find $f'(2)$, $f''(1)$, $f''(-1)$, $f'(-2)$. *Ans.* -11, -6, -18, 37.

5. In the equation $y=10\,x^6+36\,x^5-75\,x^4-300\,x^3+120\,x^2+720\,x+1$, find the value or values of x, (1) that make $f'(x)$ zero and $f''(x)$ positive; (2) that make $f'(x)$ zero and $f''(x)$ negative; (3) that make $f^{\text{IV}}(x)$ zero and $f^{\text{V}}(x)$ positive; (4) that make $f^{\text{IV}}(x)$ zero and $f^{\text{V}}(x)$ negative.

Ans. (1) -3, -1, 2; (2) -2, 1;

(3) $\dfrac{-6+\sqrt{86}}{10}$; (4) $\dfrac{-6-\sqrt{86}}{10}$.

CHAPTER VI

APPLICATIONS OF DERIVATIVES TO CURVES. MAXIMA AND MINIMA

57. In the curve whose equation is $y=f(x)$, x is the abscissa, and y or $f(x)$ the ordinate of any point (x, y) on the curve. Suppose that $f(x)$ is a single valued and continuous function of x. If x continually increases, the point (x, y) moves on the curve so that its projection on the x-axis moves from left to right. If $f(x)$ continually increases as x increases, the ordinate of the curve corresponding to the abscissa x is continually increasing as the curve is being drawn from left to right. That is, the curve is rising when drawn from left to right. If $f(x)$ continually decreases as x increases, the ordinate of the curve corresponding to the abscissa x is continually decreasing as the curve is being drawn from left to right. That is, the curve is falling when drawn from left to right. If $f(x)$ is neither increasing nor decreasing as x increases, the curve is neither rising nor falling and must therefore be parallel to the x-axis. In this case the equation becomes $y=c$, where c is constant, which is the equation of a line parallel to the x-axis.

We can thus think of the function $f(x)$ as increasing, decreasing, or remaining constant, as x increases, if the thought is merely of the function as given by the equation $y=f(x)$, or of the curve as rising, falling, or remaining parallel to the x-axis, when drawn from left to right, if the thought is of the curve that represents the equation $y=f(x)$.

58. Let us consider the curve whose equation is $y=x^3-4x^2+4x-1$ and determine the values of x between which the

curve is rising or falling, or where it is turning from rising to falling or from falling to rising, when drawn from left to right.

Let $x = x_0$. To determine whether the curve is rising or falling as x increases through x_0, we shall calculate the ordinates corresponding to x_0 and $x_0 + \Delta x$, and compare the two in length.

Let $x = x_0$. $\therefore\ y_0 = x_0^3 - 4x_0^2 + 4x_0 - 1.$

Let $x = x_0 + \Delta x$.

$$\therefore\ y_0 + \Delta y = (x_0 + \Delta x)^3 - 4(x_0 + \Delta x)^2 + 4(x_0 + \Delta x) - 1.$$

$$\therefore\ \Delta y = (3x_0^2 - 8x_0 + 4)\Delta x + (3x_0 - 4)\overline{\Delta x}^2 + \overline{\Delta x}^3.$$

$$\therefore\ \frac{\Delta y}{\Delta x} = (3x_0^2 - 8x_0 + 4) + (3x_0 - 4)\Delta x + \overline{\Delta x}^2.$$

By taking Δx small enough, $\frac{\Delta y}{\Delta x}$ can be made to have a value as near $3x_0^2 - 8x_0 + 4$ as we please. Therefore, if Δx is small enough, $\frac{\Delta y}{\Delta x}$ will have the same sign as $3x_0^2 - 8x_0 + 4$. Suppose that $3x_0^2 - 8x_0 + 4$ is positive. Therefore, if Δx is small enough, $\frac{\Delta y}{\Delta x}$ is positive. Now Δx is positive since by supposition the variable is increasing. Therefore Δy is positive. Therefore $y_0 + \Delta y$ is greater than y_0. Therefore, as Δx increases from zero, or as x increases from x_0, the ordinate of the curve is increasing. That is, the curve is rising. Therefore if x_0 is such that $3x_0^2 - 8x_0 + 4$ is positive, the curve is rising at $x = x_0$, when drawn from left to right.

The student can readily make the necessary changes in the proof and show that if $3x_0^2 - 8x_0 + 4$ is negative, the curve is falling at $x = x_0$, when drawn from left to right.

The values of x between which $3x_0^2 - 8x_0 + 4$ is positive or negative can readily be found by factoring the expression $3x_0^2 - 8x_0 + 4$. The factors are $3x_0 - 2$ and $x_0 - 2$.

If x_0 is less than $\frac{2}{3}$, both factors are negative, and therefore their product is positive. If x_0 is greater than $\frac{2}{3}$ and less than

2, $3x_0 - 2$ is positive and $x_0 - 2$ is negative, and therefore their product is negative. If x_0 is greater than 2, both factors are positive, and therefore their product is positive. The curve is therefore:

rising for all values of x less than $\frac{2}{3}$;
falling for all values of x between $\frac{2}{3}$ and 2;
rising for all values of x greater than 2.

The curve, when drawn from left to right, is turning at $x = \frac{2}{3}$ from rising to falling, and at $x = 2$ from falling to rising. At these values $3x_0^2 - 8x_0 + 4$ is zero.

59. In the equation we have been considering, namely,

$$y = x^3 - 4x^2 + 4x - 1,$$

$$(3x_0^2 - 8x_0 + 4) + (3x_0 - 4)\Delta x + \overline{\Delta x}^2 = \frac{\Delta y}{\Delta x} \text{ when } x = x_0.$$

$$\therefore\ 3x_0^2 - 8x_0 + 4 = \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta x}\right] \text{ when } x = x_0,$$

$$= \left.\frac{dy}{dx}\right|_{x=x_0}.$$

We therefore see that in this particular problem as x increases through the value x_0, if $\left.\frac{dy}{dx}\right|_{x=x_0}$ is positive the curve is rising at $x = x_0$, and if $\left.\frac{dy}{dx}\right|_{x=x_0}$ is negative, the curve is falling at $x = x_0$.

60. In general, in any equation $y = f(x)$ where $f(x)$ is a single valued and continuous function of x, as x increases through x_0, if $\left.\frac{dy}{dx}\right|_{x=x_0}$ is positive, the curve that represents the equation is rising, and if $\left.\frac{dy}{dx}\right|_{x=x_0}$ is negative, the curve is falling when $x = x_0$.

This theorem can be readily established as follows:

From the definition of a limit the difference between a function and its limit is infinitesimal. Therefore, since $\frac{\Delta y}{\Delta x}$ when $x = x_0$ is a variable, and $\left.\frac{dy}{dx}\right|_{x=x_0}$ its limit,

$$\frac{\Delta y}{\Delta x} - \left.\frac{dy}{dx}\right|_{x=x_0} = \epsilon, \text{ where } \epsilon \text{ is infinitesimal.}$$

Suppose that $\left.\dfrac{dy}{dx}\right|_{x=x_0}$ is not zero.

By taking Δx small enough, ϵ can be made to have a value as near zero as we please. Therefore, if Δx is small enough, $\dfrac{\Delta y}{\Delta x}$ when $x = x_0$ will have the same sign as $\left.\dfrac{dy}{dx}\right|_{x=x_0}$.

Suppose that $\left.\dfrac{dy}{dx}\right|_{x=x_0}$ is positive.

If Δx is small enough, $\dfrac{\Delta y}{\Delta x}$ when $x = x_0$ is positive. Now Δx is positive since by supposition the variable is increasing. Therefore Δy is positive. Since Δy is positive, $y_0 + \Delta y$ is greater than y_0. Therefore as Δx increases from zero, or as x increases from x_0, the ordinate of the curve is increasing. Therefore at $x = x_0$ the curve when drawn from left to right is rising.

Suppose that $\left.\dfrac{dy}{dx}\right|_{x=x_0}$ is negative.

If Δx is small enough, $\dfrac{\Delta y}{\Delta x}$ when $x = x_0$ is negative. Now Δx is positive since by supposition the variable is increasing. Therefore Δy is negative. Since Δy is negative, $y_0 + \Delta y$ is less than y_0. Therefore as Δx increases from zero or as x increases from x_0, the ordinate of the curve is decreasing. Therefore at $x = x_0$ the curve when drawn from left to right is falling.

61. When the curve is drawn from left to right, the slope of the tangent line may be increasing or decreasing while either positive or negative.

If it is increasing and positive, the tangent line will take a succession of positions in the order 1, 2, 3, 4 of Fig. 15, and

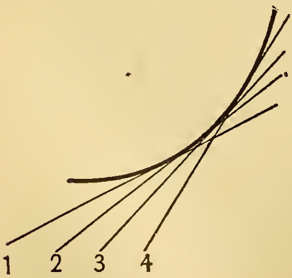


Fig. 15.

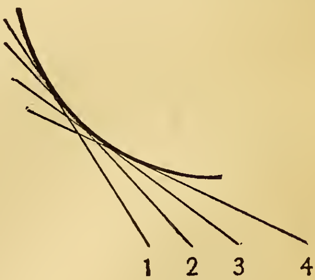


Fig. 16.

the curve will be as drawn in Fig. 15. If it is increasing and negative, the tangent line will take a succession of positions in the order 1, 2, 3, 4 of Fig. 16, and the curve will be as drawn in Fig. 16. In either case the curve is said to be **concave upwards.**

If the slope of the tangent line is decreasing and positive, the tangent line will take a succession of positions in the order 1, 2, 3, 4 of Fig. 17, and the curve will be drawn as in Fig. 17.

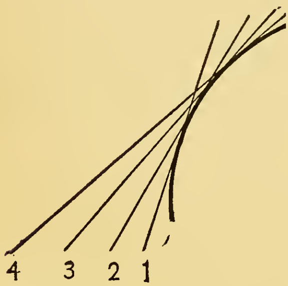


FIG. 17.

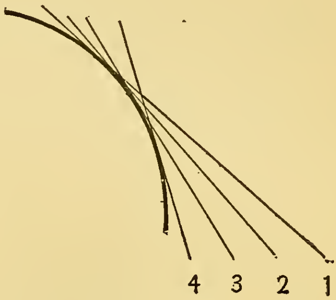


FIG. 18.

If it is decreasing and negative, the tangent line will take a succession of positions in the order 1, 2, 3, 4 of Fig. 18, and the curve will be as drawn as in Fig. 18. In either case the curve is said to be **concave downwards.**

62. In Art. 58, we found a method for determining the values of x between which the curve $y = x^3 - 4x^2 + 4x - 1$ is rising or falling or for which it is neither rising nor falling, when drawn from left to right. We shall now find a method for determining the values of x in the same curve between which the curve is concave upwards or concave downwards or for which it is neither concave upwards nor concave downwards.

If $$y = x^3 - 4x^2 + 4x - 1,$$

then $$\frac{dy}{dx} = 3x^2 - 8x + 4,$$

and $$\frac{d^2y}{dx^2} = 6x - 8.$$

The expression $3x^2 - 8x + 4$ is a function of x. Call it u. Then if $\frac{du}{dx}$ is positive, the curve whose equation is $u = 3x^2 - 8x + 4$ is rising; that is, u is increasing. (See Art. 60.) Now $\frac{du}{dx}$ is $\frac{d^2y}{dx^2}$. Therefore if $\frac{d^2y}{dx^2}$ is positive, u is increasing. But u, or $\frac{dy}{dx}$, is the slope of the tangent line to the curve at a point (x, y) on the curve. (See Art. 39.) Therefore, if $\frac{d^2y}{dx^2}$ is positive, the slope of the tangent line to the curve at a point on the curve is increasing. Therefore the curve is concave upwards.

A similar proof will show that if $\frac{d^2y}{dx^2}$ is negative, the curve is concave downwards.

The expression $6x - 8$ is negative if x is less than $\frac{4}{3}$ and positive if x is greater than $\frac{4}{3}$. The curve is therefore concave downwards for all values of x less than $\frac{4}{3}$ and concave upwards for all values of x greater than $\frac{4}{3}$.

63. We saw in Art. 58 that, in the curve $y = x^3 - 4x^2 + 4x - 1$, for values of x:

less than $\frac{2}{3}$, the curve is rising;
between $\frac{2}{3}$ and 2, the curve is falling;
greater than 2, the curve is rising;

and in the preceding article that, for values of x:

less than $\frac{4}{3}$, the curve is concave downwards;
greater than $\frac{4}{3}$, the curve is concave upwards.

From these results, we can plot the curve by plotting a few points and drawing the curve through them to satisfy the above data.

The ordinates whose abscissas are $\frac{2}{3}$, $\frac{4}{3}$, and 2 are $\frac{5}{27}$, $-\frac{11}{27}$, and -1 respectively. Plot these points. They are D, F, and G in the figure. When $x = 0$, $y = -1$. When $y = 0$, $x = \frac{3 - \sqrt{5}}{2}$, 1, and $\frac{3 + \sqrt{5}}{2}$. Plot these points. They are B, C, E, and H

in the figure. When $x=-1$, $y=-10$. When $x=3$, $y=2$. Plot these points. They are A and K in the figure. Draw a curve concave downwards through $ABCDEF$, then concave upwards through GHK. We thus have the curve drawn between $x=-1$ and $x=3$.

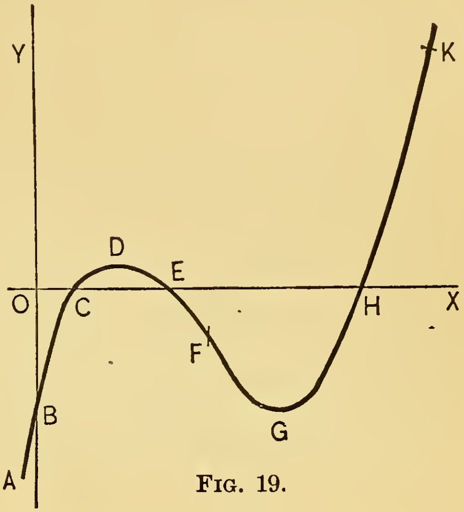


FIG. 19.

64. In Art. 62, we saw that, in the curve $y=x^3-4x^2+4x-1$, if $\frac{d^2y}{dx^2}$ is positive, the curve is concave upwards, and if negative, concave downwards. That this is true for any curve $y=f(x)$, where $f(x)$ is a single valued and continuous function of x, can readily be shown.

In the equation $y=f(x)$, $\frac{dy}{dx}$ is a function of x. Then if its derivative, namely $\frac{d^2y}{dx^2}$, is positive, $\frac{dy}{dx}$ is increasing and the curve is concave upwards. If $\frac{d^2y}{dx^2}$ is negative, $\frac{dy}{dx}$ is decreasing and the curve is concave downwards.

65. **Points of inflexion.** The points on the curve at which the curve turns from concavity upwards to concavity downwards or vice versa are called **points of inflexion** of the curve.

Thus, in the curve $y=x^3-4x^2+4x-1$, the point $(\frac{4}{3}, -\frac{11}{27})$ is a point of inflexion.

66. In Art. 58, we saw that in the curve $y=x^3-4x^2+4x-1$, when $\left.\frac{dy}{dx}\right|_{x=x_0}$ or $3x_0^2-8x_0+4$ is zero, the curve is turning from rising to falling or from falling to rising. It is not true, however, that in any curve $y=f(x)$ when $\left.\frac{dy}{dx}\right|_{x=x_0}$ is zero the curve is turning

from rising to falling or from falling to rising. It may be, as in Fig. 20 or Fig. 21, where it is rising or falling, and at a par-

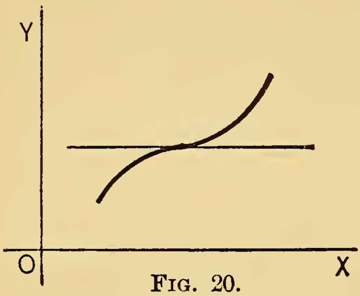


FIG. 20.

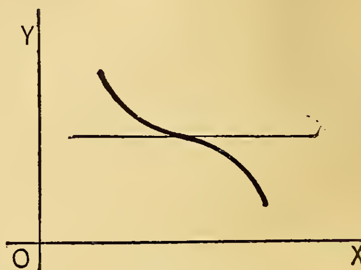


FIG. 21.

ticular point $x = x_0$ has a value at which the tangent line is parallel to the x-axis.

As an illustration, consider the curve

$$y = x^4 - 6x^2 + 8x + 1.$$

$$\frac{dy}{dx} = 4x^3 - 12x^2 + 8.$$

$$\frac{d^2y}{dx^2} = 12(x^2 - 1).$$

For values of x in the neighborhood of 1, $\frac{dy}{dx}$ is positive, and therefore for these values the curve is rising. When $x = 1$, $\frac{dy}{dx}$ is zero, and therefore the tangent line to the curve at this point is parallel to the x-axis. For values of x immediately less than 1, $\frac{d^2y}{dx^2}$ is negative, and therefore for these values the curve is concave downwards. For values of x immediately greater than 1, $\frac{d^2y}{dx^2}$ is positive, and therefore for these values the curve is concave upwards. In the neighborhood of $x = 1$ it is therefore as in Fig. 22.

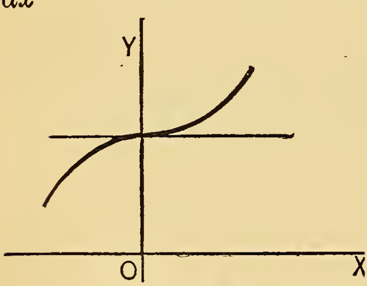


FIG. 22.

67. Definitions. If a curve rises to a certain point and then falls, that point is called a **maximum point** of the curve.

If a curve falls to a certain point and then rises, that point is called a **minimum point** of the curve.

Thus, in the curve $y = x^3 - 4x^2 + 4x - 1$, $(\frac{2}{3}, \frac{5}{27})$ is a maximum point and $(2, -1)$ is a minimum point of the curve.

68. We saw that, if $\frac{dy}{dx}$ is positive, the curve is rising, and if negative, falling when drawn from left to right. Then, at a maximum point, $\frac{dy}{dx}$ changes from positive to negative. Now a continuous function can change in sign only by passing through the value zero, and a function which has only infinite discontinuities, by passing through the value zero or by becoming infinite. Therefore, at a maximum point, $\frac{dy}{dx}$ is zero or

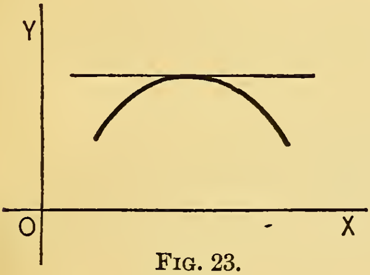


FIG. 23.

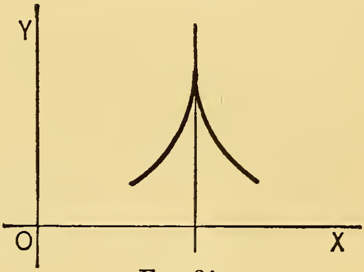


FIG. 24.

infinite. The curve at a maximum point is therefore shaped as in Fig. 23, where the tangent line is parallel to the x-axis, or as in Fig. 24, where it is perpendicular to the x-axis.

The maximum point at which $\frac{dy}{dx}$ is zero may be called an ordinary maximum point. It is the kind that appears in the curve of Art. 63. The maximum point at which $\frac{dy}{dx}$ is infinite is called a **cusp maximum** point.

Similarly we may have an ordinary minimum point of the kind that appears in Art. 63, and a **cusp minimum** point at which $\frac{dy}{dx}$ is infinite. In the latter case the curve appears at the point, as in Fig. 25.

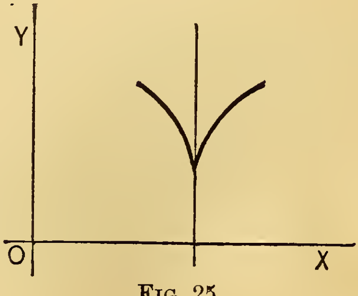


FIG. 25.

Tests for Maximum or Minimum Values

69. We saw that, if $\frac{dy}{dx}$ is zero for $x = x_0$, and if $\frac{dy}{dx}$ is positive before $x = x_0$ and negative after $x = x_0$, the point (x_0, y_0) on the curve is a maximum point.

In this condition for a maximum point, the conditions that $\frac{dy}{dx}$ is positive before $x = x_0$ and negative after $x = x_0$ may be expressed in the one condition, $\left.\frac{d^2y}{dx^2}\right|_{x=x_0}$ is negative.

For, suppose that $\frac{d^2y}{dx^2}$ is negative when $x = x_0$. Since $\frac{d^2y}{dx^2}$, the derivative with respect to x of $\frac{dy}{dx}$, is negative when $x = x_0$, $\frac{dy}{dx}$ is decreasing as x passes through the value x_0, and since $\frac{dy}{dx}$ is zero when $x = x_0$, it must have been positive before $x = x_0$ and negative after $x = x_0$.

We can therefore conclude that, if

$$\left.\frac{dy}{dx}\right|_{x=x_0} = 0,$$

and

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = -,$$

the point (x_0, y_0) on the curve is a maximum point.

Similarly, if

$$\left.\frac{dy}{dx}\right|_{x=x_0} = 0,$$

and

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = +,$$

the point (x_0, y_0) on the curve is a minimum point.

70. The above reasoning is based on the supposition that the function $f(x)$ is single-valued and continuous for all the values of x considered.

If the function $f(x)$ is not single-valued it can be made so, as explained in Art. 15.

If the function is infinite or otherwise discontinuous for certain values of x, these values may be avoided in the discussion. For all other values the reasoning of the preceding articles will hold.

71. As an illustration of the method of plotting a curve when the function which it represents is infinite for certain values of the variable, consider the curve

$$y = \frac{x(x-1)}{2x-3}.$$

For $x = \frac{3}{2}$, the function is infinite or infinite negatively, so that this value of x will be avoided in the discussion. For all other values of x,

$$\frac{dy}{dx} = \frac{2x^2 - 6x + 3}{(2x-3)^2},$$

and

$$\frac{d^2y}{dx^2} = \frac{6}{(2x-3)^3}.$$

Factor the numerator of $\dfrac{2x^2 - 6x + 3}{(2x-3)^2}$.

$$\therefore\ \frac{dy}{dx} = \frac{2\left\{x - \dfrac{3-\sqrt{3}}{2}\right\}\left\{x - \dfrac{3+\sqrt{3}}{2}\right\}}{(2x-3)^2}.$$

An examination of this expression shows that $\dfrac{dy}{dx}$ is:

positive for all values of x less than $\dfrac{3-\sqrt{3}}{2}$;

negative for all values of x between $\dfrac{3-\sqrt{3}}{2}$ and $\dfrac{3+\sqrt{3}}{2}$, avoiding $\dfrac{3}{2}$;

positive for all values of x greater than $\dfrac{3+\sqrt{3}}{2}$; and

zero when $x = \dfrac{3-\sqrt{3}}{2}$ and $x = \dfrac{3+\sqrt{3}}{2}$.

The curve is therefore:

rising for all values of x less than $\frac{3-\sqrt{3}}{2}$;

falling for all values of x between $\frac{3-\sqrt{3}}{2}$ and $\frac{3+\sqrt{3}}{2}$, avoiding $\frac{3}{2}$;

rising for all values of x greater than $\frac{3+\sqrt{3}}{2}$.

An examination of $\frac{6}{(2x-3)^3}$ shows that $\frac{d^2y}{dx^2}$ is:

negative for all values of x less than $\frac{3}{2}$;

positive for all values of x greater than $\frac{3}{2}$.

The curve is therefore:

concave downwards for all values of x less than $\frac{3}{2}$;

concave upwards for all values of x greater than $\frac{3}{2}$;

and has a maximum point when $x=\frac{3-\sqrt{3}}{2}$, and a minimum point when $x=\frac{3+\sqrt{3}}{2}$.

An examination of $\frac{x(x-1)}{2x-3}$ shows that the curve cuts the x-axis when $x=0$ and $x=1$, and that

$$\operatorname*{limit}_{x=-\infty}\left[\frac{x(x-1)}{2x-3}\right]=-\infty,$$

and

$$\operatorname*{limit}_{x=\infty}\left[\frac{x(x-1)}{2x-3}\right]=+\infty.$$

The curve is therefore as drawn in Fig. 26.

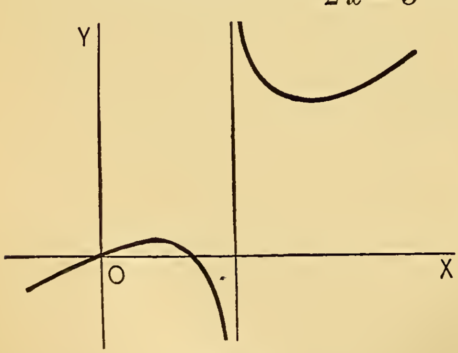


Fig. 26.

Since $\frac{d^2y}{dx^2}$ cannot become zero for any value of x, this curve has no point of inflection.

EXERCISES

1. Draw a branch of a curve at every point of which:

(*a*) x is positive, y is positive, $\frac{dy}{dx}$ is positive, and $\frac{d^2y}{dx^2}$ is positive;

(*b*) x is positive, y is negative, $\frac{dy}{dx}$ is positive, and $\frac{d^2y}{dx^2}$ is positive;

(*c*) x is negative, y is positive, $\frac{dy}{dx}$ is negative, and $\frac{d^2y}{dx^2}$ is negative;

(*d*) x is negative, y is positive, $\frac{dy}{dx}$ is negative, and $\frac{d^2y}{dx^2}$ is positive.

2. In the curve $y=f(x)$, show that if $\left.\frac{d^2y}{dx^2}\right|_{x=x_0}=0$, and $\left.\frac{d^3y}{dx^3}\right|_{x=x_0}=+$, the curve has a point of inflection when $x=x_0$.

3. In Exercise 2, if $\left.\frac{dy}{dx}\right|_{x=x_0}=0$ also, how is the curve situated at the point of inflection?

Ans. The curve is parallel to the x-axis.

4. In the curve $y=f(x)$, if $\left.\frac{dy}{dx}\right|_{x=x_0}=0$, $\left.\frac{d^2y}{dx^2}\right|_{x=x_0}=0$, $\left.\frac{d^3y}{dx^3}\right|_{x=x_0}=0$, and $\left.\frac{d^4y}{dx^4}\right|_{x=x_0}=+$, show that the curve has a minimum point when $x=x_0$.

Investigate the following curves for ranges of values of x in which the curves are: (1) rising; (2) falling; (3) concave upwards; (4) concave downwards; and for values of x for which they have: (5) maximum points; (6) minimum points; (7) points of inflection. In each case, plot the curve.

5. $y=x^2-2x+4$.

6. $y=x(x-1)(x-2)$.

7. $y=(x-1)(x-2)(x-3)$.

8. $y=x(x^2-3)$.

9. $y=x^2(x^2-3)$.

10. $y=x^3$.

11. $y=x^4$.

12. $y=x(x-3)^3$.

13. $y=x(x^2+1)$.

14. $y=x^3(x-3)$.

15. $y = x + \frac{1}{x}$.

16. $y = \frac{x}{x+1}$.

17. $y = \frac{x-1}{x+2}$.

18. $y = \frac{x^2}{x+1}$.

19. $y = \frac{x(x+1)}{x-2}$.

20. $y = \frac{(x-1)(x-2)}{x+1}$.

Investigate the following curves for maximum and minimum points. In each case plot the curve.

21. $y^2 = 4x(1-x)$.

22. $y^2 = x(x^2-1)$.

23. $y^2 = x^3(1-x)$.

24. $y^2 = x^3$.

25. $y^2 = \frac{1}{x}$.

26. $y^3 = x + 2$.

27. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

28. $y = (x-1)^{\frac{2}{3}} + 4$.

29. $y = (x-1)^{\frac{1}{3}} + 3$.

72. **Definitions.** If a single valued and continuous function increases to a certain value and then decreases, that value is called a **maximum** value of the function.

If a single valued and continuous function decreases to a certain value and then increases, that value is called a **minimum** value of the function.

73. We saw in Art. 57 that if the curve $y = f(x)$ is rising, the function $f(x)$ is increasing, and if falling, decreasing. The tests for a maximum or minimum value of the function will therefore be the same as for a maximum or minimum point of the curve, namely:

If
$$\left.\frac{dy}{dx}\right|_{x=x_0} = 0,$$

and
$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = -,$$

as x increases through the value x_0, the function $f(x)$ has a maximum value when $x = x_0$; and

If
$$\left.\frac{dy}{dx}\right|_{x=x_0} = 0,$$
and
$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = +,$$
as x increases through the value x_0, the function $f(x)$ has a minimum value when $x = x_0$.

74. As an illustration of some of the uses to which the above tests for maximum or minimum values of a function may be put, consider the following example:

EXAMPLE. Given that the strength of a beam of rectangular cross section is proportional to the product of its breadth by the square of its depth, find the breadth and depth of the strongest rectangular beam that can be cut from a cylindrical log which is $2a$ inches in diameter.

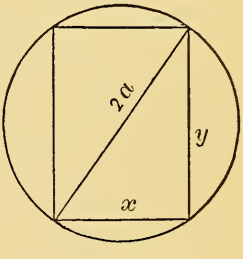


FIG. 27.

Let s, x, and y denote the strength, breadth, and depth respectively of the beam.

Then by supposition, $s \propto xy^2$. $\therefore s = kxy^2$ where k is a constant depending on the material of which the beam is composed.

From the geometry of the figure, $x^2 + y^2 = 4a^2$.

$$\therefore\ s = kx(4a^2 - x^2).$$

$$\therefore\ \frac{ds}{dx} = k\{4a^2 - 3x^2\}.$$

$$\therefore\ \frac{d^2s}{dx^2} = -6kx.$$

When $x = \dfrac{2a}{\sqrt{3}}$, $\dfrac{ds}{dx} = 0$ and $\dfrac{d^2s}{dx^2} = -$. Therefore, when $x = \dfrac{2a}{\sqrt{3}}$, the strength of the beam is a maximum. Substitute $x = \dfrac{2a}{\sqrt{3}}$ in $x^2 + y^2 = 4a^2$. Therefore $y = 2a\sqrt{\tfrac{2}{3}}$. Therefore the dimensions of the strongest beam are: breadth $\dfrac{2a}{\sqrt{3}}$, depth $2a\sqrt{\tfrac{2}{3}}$.

EXERCISES

1. Given that the stiffness of a beam of rectangular cross section is proportional to the product of its breadth and the cube of its depth, find the breadth and depth of the beam of greatest stiffness that can be cut from a cylindrical log $2a$ inches in diameter.

Ans. Breadth $= a$ in. Depth $= a\sqrt{3}$ in.

2. A rectangular box with open top and square base is to be constructed to hold 2 gallons. Find the dimensions of the box in inches which will contain the least possible material. (231 cu. in. = 1 gal.)

Ans. Side of base = 9.74 in. Height = 4.87 in.

3. A rectangular box with closed top and square base is to be constructed to hold 2 bushels. The cost of the material in the bottom is 3 cents per square inch, in the ends 2 cents per square inch, and in the sides and top 1 cent per square inch. Find the dimensions of the box of least cost. (2150.42 cu. in. = 1 bu.)

Ans. Side of base = 14.77 in. Height = 19.71 in.

4. A wall 6 feet high is situated 8 feet from the side of a house. What is the length of the shortest ladder that will reach from the ground beyond the wall to the house?

Ans. 19.73 ft.

5. Show that the maximum rectangle inscriptable in a circle is a square.

6. Find the right circular cone of maximum volume that can be inscribed in a sphere of radius R.

Ans. Radius of base $= \frac{2}{3}\sqrt{2}\,R$. Height $= \frac{4}{3}R$.

7. Find the altitude of the right circular cylinder of maximum surface that can be inscribed in a sphere whose radius is R.

Ans. $\left(2 - \dfrac{2}{\sqrt{5}}\right)^{\frac{1}{2}} R.$

8. Find the altitude of the right circular cylinder of greatest volume that can be inscribed in a sphere whose radius is R.

Ans. $\frac{2R}{\sqrt{3}}$.

9. Find the altitude and volume of the right circular cone of least volume that can be circumscribed about a sphere whose radius is R. *Ans.* Altitude $= 4R$. Volume $= \frac{8}{3}\pi R^3$.

10. A square piece of tin each of whose sides is a has a small square cut out of each corner. Find the side of the small square that the remainder may form a box of maximum contents.

Ans. $\frac{a}{6}$.

11. The work of propelling a steamer through the water varies as the cube of her speed. Find the most economical speed against a current running 4 miles per hour.

Ans. 6 mi. per hr.

12. The cost of fuel consumed in propelling a steamer through the water varies as the cube of her speed, and is \$25 per hour when the speed is 10 miles per hour. The other expenses are \$100 per hour. Find the most economical speed. *Ans.* 12.6 mi. per hr.

13. A Norman window consists of a rectangle surmounted by a semicircle. For a given perimeter, determine the breadth and height of the window that gives a maximum quantity of light.

Ans. Breadth = height of rectangle $= \frac{2p}{\pi + 4}$, where p is the perimeter.

14. The area cut off from a parabola by any double ordinate is $\frac{2}{3}$ the circumscribing rectangle. Find the maximum parabola which can be cut off from a right cone, in terms of the slant height of the cone and the radius of the base.

Ans. $\frac{1}{2}ha\sqrt{3}$, where a = radius of base and h = slant height.

15. A barnyard gate is 400 feet from a straight river. The gate of the pasture is 100 feet from the river and 500 feet from the barnyard gate. If the cattle go to drink on their way to the pasture, determine the length of their shortest possible path from gate to gate. *Ans.* 640.31 ft.

16. A miner wishes to dig a tunnel from a point A to a point B, 300 feet below A and 500 feet distant from it. Below A is bed rock and above A is earth, the separating surface being a horizontal plane. Given that the cost of tunneling through earth is \$1 and through rock \$3 per linear foot, find the cost of a tunnel of minimum cost. *Ans.* \$1248.53.

17. A man's house is situated 20 yards from a straight car track. The man can walk at the rate of 4 miles per hour and the car travels at the rate of 8 miles per hour. Where must the man leave the car in order that he may reach home in the least possible time.

Ans. 34.64 ft. from foot of perpendicular from house to track.

18. Assuming that the current in a voltaic cell is $C = \frac{E}{r+R}$, E being the electromotive force, r the internal resistance, R the external resistance; and that the power given out is $P = RC^2$; prove that P is a maximum when $r = R$. Trace the curve that shows the variation of P as R varies.

(Perry's *Calculus for Engineers.*)

19. A length l of wire is to be cut into two portions, which are to be bent into the forms of a circle and square respectively. Show that the sum of the areas of these figures will be least when the wire is cut in the ratio $\pi : 4$.

CHAPTER VII

FORMULAS FOR DIFFERENTIATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS. HYPERBOLIC FUNCTIONS

75. The formulas of Art. 52 are not sufficient to enable us to determine the derivative of any other than an algebraic function, as can readily be seen by an example.

Example. Suppose that it is required to find $\frac{dy}{dx}$, if

$$y = x^2 \log_{10}(x^3+1).$$

$$\frac{dy}{dx} = \frac{x^2 d\{\log_{10}(x^3+1)\}}{dx} + \{\log_{10}(x^3+1)\}\frac{dx^2}{dx}, \text{ by Formula IV},$$

$$= \frac{x^2 d\{\log_{10}(x^3+1)\}}{dx} + 2\,x \log_{10}(x^3+1), \text{ by Formula VII.}$$

Since none of the formulas of Art. 52 gives a means of finding $\frac{d \log_{10}(x^3+1)}{dx}$, $\frac{dy}{dx}$ is not determined.

76. The following formulas with those of Art. 52 will enable us to find the derivative of any logarithmic or exponential function.

$$\text{VIII.} \quad \frac{d \log_a u}{dx} = \frac{1}{u}\frac{du}{dx} \log_a e.$$

$$\text{IX.} \quad \frac{d \log_e u}{dx} = \frac{1}{u}\frac{du}{dx}.$$

$$\text{X.} \quad \frac{da^u}{dx} = a^u \frac{du}{dx} \log_e a.$$

$$\text{XI.} \quad \frac{de^u}{dx} = e^u \frac{du}{dx}.$$

$$\text{XII.} \quad \frac{du^v}{dx} = vu^{v-1}\frac{du}{dx} + u^v \frac{dv}{dx} \log_e u.$$

In these formulas, u and v are functions of x; a is any constant capable of being a base to a system of logarithms; and e is a particular constant whose value to seven places of decimals is calculated further on and found to be $2.7182818 \cdots$.

77. Derivation of formulas.

Proof of VIII.

Let $\qquad y = \log_a u.$

Let $x = x_0$. $\qquad \therefore\ y_0 = \log_a u_0.$

Let $x = x_0 + \Delta x$. $\qquad \therefore\ y_0 + \Delta y = \log_a (u_0 + \Delta u).$

Subtract.

$$\therefore\ \Delta y = \log_a (u_0 + \Delta u) - \log_a u_0$$

$$= \log_a \frac{u_0 + \Delta u}{u_0}$$

$$= \log_a \left(1 + \frac{\Delta u}{u_0}\right).$$

Divide by Δx.

$$\therefore\ \frac{\Delta y}{\Delta x} = \frac{\log_a \left(1 + \frac{\Delta u}{u_0}\right)}{\Delta x}$$

$$= \frac{1}{u_0} \cdot \frac{\Delta u}{\Delta x} \cdot \frac{u_0}{\Delta u} \cdot \log_a \left(1 + \frac{\Delta u}{u_0}\right)$$

identically,

$$= \frac{1}{u_0} \cdot \frac{\Delta u}{\Delta x} \cdot \log_a \left(1 + \frac{\Delta u}{u_0}\right)^{\frac{u_0}{\Delta u}},$$

since $m \log_a N = \log_a N^m$.

$$\therefore\ \left.\frac{dy}{dx}\right|_{x=x_0} = \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta y}{\Delta x}\right]$$

$$= \frac{1}{u_0} \cdot \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta u}{\Delta x}\right] \cdot \underset{\Delta x \doteq 0}{\text{limit}} \left[\log_a \left(1 + \frac{\Delta u}{u_0}\right)^{\frac{u_0}{\Delta u}}\right]$$

$$= \frac{1}{u_0} \left.\frac{du}{dx}\right|_{x=x_0} \cdot \underset{\Delta x \doteq 0}{\text{limit}} \left[\log_a \left(1 + \frac{\Delta u}{u_0}\right)^{\frac{u_0}{\Delta u}}\right].$$

It remains to find $\underset{\Delta x \doteq 0}{\text{limit}} \left[\log_a \left(1 + \frac{\Delta u}{u_0}\right)^{\frac{u_0}{\Delta u}}\right].$

Let $\dfrac{u_0}{\Delta u} = z$. As $\Delta x \doteq 0$, $\Delta u \doteq 0$, and $\dfrac{u_0}{\Delta u}$, or z, $= \infty$.

$$\therefore \ \underset{\Delta x \doteq 0}{\text{limit}}\left[\log_a\left(1 + \frac{\Delta u}{u_0}\right)^{\frac{u_0}{\Delta u}}\right] = \underset{z=\infty}{\text{limit}}\left[\log_a\left(1 + \frac{1}{z}\right)^z\right].$$

To find $\underset{z=\infty}{\text{limit}}\left[\log_a\left(1+\dfrac{1}{z}\right)^z\right]$, expand $\left(1+\dfrac{1}{z}\right)^z$ by the Binomial Theorem, assuming z to be a large positive integer.

$$\left(1+\frac{1}{z}\right)^z = 1 + z\,\frac{1}{z} + \frac{z(z-1)}{\underline{|2}}\left(\frac{1}{z}\right)^2 + \frac{z(z-1)(z-2)}{\underline{|3}}\left(\frac{1}{z}\right)^3 + \cdots + \left(\frac{1}{z}\right)^z$$

$$= 1 + 1 + \frac{1-\frac{1}{z}}{\underline{|2}} + \frac{\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)}{\underline{|3}} + \cdots + \left(\frac{1}{z}\right)^z.$$

Now $\underset{z=\infty}{\text{limit}}\left[\left(1+\dfrac{1}{z}\right)^z\right]$

$$= \underset{z=\infty}{\text{limit}}\left[1 + 1 + \frac{1-\frac{1}{z}}{\underline{|2}} + \frac{\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)}{\underline{|3}} + \cdots + \left(\frac{1}{z}\right)^z\right]$$

$$= \text{the infinite series } 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots.$$

Call the value of this series e.

$$\therefore \ \left.\frac{dy}{dx}\right|_{x=x_0} = \left.\frac{1}{u_0}\frac{du}{dx}\right|_{x=x_0} \log_a e.$$

$$\therefore \ \frac{d \log_a u}{dx} = \frac{1}{u}\frac{du}{dx}\log_a e.$$

Note. In showing that $\underset{z=\infty}{\text{limit}}\left[\log_a\left(1+\dfrac{1}{z}\right)^z\right] = \log_a e$, a number of assumptions were made that would require closer investigation for a rigorous proof. For a rigorous proof the student is referred to Byerly's *Differential Calculus*, pages 50, 51, and 52.

To calculate the value of e from the infinite series, to seven decimal places:

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \frac{1}{\underline{|5}} + \frac{1}{\underline{|6}} + \frac{1}{\underline{|7}} + \frac{1}{\underline{|8}} + \frac{1}{\underline{|9}} + \frac{1}{\underline{|10}} + \frac{1}{\underline{|11}} + \cdots$$

$$\begin{aligned}
= \quad & 1.0 \\
& + 1.0 \\
& + 0.5 \\
& + 0.166666666 \cdots \\
& + 0.041666666 \cdots \\
& + 0.008333333 \cdots \\
& + 0.001388888 \cdots \\
& + 0.000198412 \cdots \\
& + 0.000024801 \cdots \\
& + 0.000002755 \cdots \\
& + 0.000000275 \cdots \\
& + 0.000000025 \cdots \\
= \quad & 2.7182818 \cdots
\end{aligned}$$

Proof of IX.

This is a special case of VIII, in which $a = e$. In this case

$$\frac{d \log_e u}{dx} = \frac{1}{u}\frac{du}{dx}, \text{ since } \log_e e = 1.$$

Logarithms to the base e are called **natural**, or sometimes **hyperbolic**, logarithms. When the base of a logarithm is not given, it is assumed that the logarithm is to the base e.

Proof of X.

Let $$y = a^u.$$

Take the logarithm of each member of the equation.

$$\begin{aligned}
\therefore \log_e y &= \log_e a^u \\
&= u \log_e a.
\end{aligned}$$

Differentiate each member of the equation with respect to x.

$$\therefore \ \frac{1}{y}\frac{dy}{dx}=\frac{du}{dx}\log_e a.$$

$$\therefore \ \frac{dy}{dx}=y\frac{du}{dx}\log_e a$$

$$=a^u\frac{du}{dx}\log_e a.$$

Proof of XI.

This is a special case of X, in which $a=e$. In this case

$$\frac{de^u}{dx}=e^u\frac{du}{dx}, \text{ since } \log_e e=1.$$

Proof of XII.

Let $$y=u^v.$$

Take the logarithm of each member of the equation.

$$\therefore \ \log_e y=v\log_e u.$$

Differentiate each member of the equation with respect to x.

$$\therefore \ \frac{1}{y}\frac{dy}{dx}=\frac{v}{u}\frac{du}{dx}+\frac{dv}{dx}\log_e u.$$

$$\therefore \ \frac{dy}{dx}=y\left(\frac{v}{u}\frac{du}{dx}+\frac{dv}{dx}\log_e u\right)$$

$$=vu^{v-1}\frac{du}{dx}+u^v\frac{dv}{dx}\log_e u, \text{ since } y=u^v.$$

This formula can be easily remembered by noting that it can be derived by differentiating u^v, first, as if v were constant and u variable; second, as if u were constant and v variable; and then adding the two results.

EXERCISES

Find $\dfrac{dy}{dx}$ in each of the following cases:

1. $y = \log(x^2 + 2x)$. *Ans.* $\dfrac{2(x+1)}{x^2+2x}$.

2. $y = \log\sqrt{x-1}$. *Ans.* $\dfrac{1}{2(x-1)}$.

3. $y = \log(x+1)^2$. *Ans.* $\dfrac{2}{x+1}$.

4. $y = x^2 \log x$. *Ans.* $x(1 + 2\log x)$.

5. $y = x\log(x^2+1)$. *Ans.* $\dfrac{2x^2}{x^2+1} + \log(x^2+1)$.

6. $y = \log(x^2+1)\sqrt{1-x^2}$. *Ans.* $\dfrac{3x^3 - x}{x^4 - 1}$.

7. $y = e^x \log x$. *Ans.* $\dfrac{e^x}{x} + e^x \log x$.

8. $y = x^n a^x$. *Ans.* $x^{n-1}a^x(x\log a + n)$.

9. $y = x^2 e^{3x}$. *Ans.* $xe^{3x}(2 + 3x)$.

10. $y = e^{4x}\log\sqrt{1+x^2}$. *Ans.* $e^{4x}\left[2\log(1+x^2) + \dfrac{x}{1+x^2}\right]$.

11. $y = \log(e^x + e^{-x})$. *Ans.* $\dfrac{e^{2x}-1}{e^{2x}+1}$.

12. $y = \log_{10}(x^2 + 5x)$.

Ans. $M\dfrac{2x+5}{x^2+5x}$, where $M = \log_{10} e = .43429\cdots$.

13. $y = 5^{x^2+3x}$. *Ans.* $5^{x^2+3x}(2x+3)\log 5$.

14. $y = 2x^{\frac{1}{2}} - 3x^{\frac{1}{3}} + 6x^{\frac{1}{6}} - 6\log(x^{\frac{1}{6}} + 1)$. *Ans.* $\dfrac{1}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$.

15. $y = x^{\frac{1}{x}}$. *Ans.* $x^{\frac{1}{x}-2}(1 - \log x)$.

16. $y = e^{e^x}$. *Ans.* $e^x e^{e^x}$.

17. $y = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. *Ans.* $\dfrac{4\, e^{2x}}{(e^{2x}+1)^2}$.

18. $y = x^{\frac{1}{\log x}}$. *Ans.* 0.

In each of the two following exercises, find $\dfrac{dy}{dz}$, $\dfrac{dz}{dx}$, and then $\dfrac{dy}{dx}$:

19. $y = (1 + e^z + e^{2z})$, $z = \log(x^2 - 2\,x + 3)$.

Ans. $\dfrac{dy}{dx} = 2\,(x-1)\,(2\,x^2 - 4\,x + 7)$.

20. $y = \log(2\,z - z^2)$, $z = e^{x-1}$. *Ans.* $\dfrac{dy}{dx} = \dfrac{2\,(1 - e^{x-1})}{2 - e^{x-1}}$.

21. In the curve $y = \log(x^2 + 2\,x)$, find the abscissas of the points at which x and y change at the same rate.

Ans. $x = \pm\sqrt{2}$.

22. Investigate $y = \log\sqrt{\dfrac{x-1}{x+1}} + \log\sqrt{\dfrac{x^3+1}{x^3-1}}$ for maximum and minimum values.

Ans. Max. $x = -1$. Min. $x = +1$.

HYPERBOLIC FUNCTIONS

78. **Definitions.** The expression $\dfrac{e^u - e^{-u}}{2}$ is denoted by sinh u and read, "hyperbolic sine of u."

The expression $\dfrac{e^u + e^{-u}}{2}$ is denoted by cosh u and read, "hyperbolic cosine of u." Also,

$\dfrac{\sinh u}{\cosh u}$, or $\dfrac{e^u - e^{-u}}{e^u + e^{-u}}$, is denoted by tanh u. Read, "hyperbolic tangent of u."

$\dfrac{\cosh u}{\sinh u}$, or $\dfrac{e^u + e^{-u}}{e^u - e^{-u}}$, is denoted by coth u. Read, "hyperbolic cotangent of u."

$\dfrac{1}{\sinh u}$ is denoted by cosech u. Read, "hyperbolic cosecant of u."

$\dfrac{1}{\cosh u}$ is denoted by sech u. Read, "hyperbolic secant of u."

EXERCISES

1. Establish the following identities:

$$\cosh^2 u - \sinh^2 u = 1. \qquad \tanh^2 u + \operatorname{sech}^2 u = 1.$$

$$\coth^2 u - \operatorname{cosech}^2 u = 1. \qquad \sinh 2u = 2 \sinh u \cosh u.$$

$$\sinh(u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v.$$

2. If u is a function of x, prove that:

$$\frac{d \sinh u}{dx} = \cosh u \frac{du}{dx}. \qquad \frac{d \cosh u}{dx} = \sinh u \frac{du}{dx}.$$

$$\frac{d \tanh u}{dx} = \operatorname{sech}^2 u \frac{du}{dx}. \qquad \frac{d \coth u}{dx} = - \operatorname{cosech}^2 u \frac{du}{dx}.$$

$$\frac{d \operatorname{cosech} u}{dx} = - \operatorname{cosech} u \coth u \frac{du}{dx}.$$

$$\frac{d \operatorname{sech} u}{dx} = - \operatorname{sech} u \tanh u \frac{du}{dx}.$$

Definitions. If $u = \sinh y$, then $y = \sinh^{-1} u$. Similarly for the other hyperbolic functions.

3. If u is a function of x, prove that:

$$\frac{d \sinh^{-1} u}{dx} = \frac{\frac{du}{dx}}{\sqrt{u^2+1}}.$$

Proof. Let $y = \sinh^{-1} u$. $\quad \therefore \sinh y = u$.

$$\therefore \cosh y \frac{dy}{dx} = \frac{du}{dx}. \qquad \therefore \frac{dy}{dx} = \frac{\frac{du}{dx}}{\cosh y} = \frac{\frac{du}{dx}}{\sqrt{u^2+1}}.$$

$$\therefore \frac{d \sinh^{-1} u}{dx} = \frac{\frac{du}{dx}}{\sqrt{u^2+1}}.$$

4. If u is a function of x, prove that:

$$\frac{d\cosh^{-1}u}{dx}=\frac{\frac{du}{dx}}{\sqrt{u^2-1}}.\qquad \frac{d\tanh^{-1}u}{dx}=\frac{\frac{du}{dx}}{1-u^2}.$$

$$\frac{d\coth^{-1}u}{dx}=\frac{\frac{du}{dx}}{1-u^2}.\qquad \frac{d\,\mathrm{cosech}^{-1}u}{dx}=-\frac{\frac{du}{dx}}{u\sqrt{1+u^2}}.$$

$$\frac{d\,\mathrm{sech}^{-1}u}{dx}=-\frac{\frac{du}{dx}}{u\sqrt{1-u^2}}.$$

5. Show that:

$$\sinh^{-1}u=\log(u+\sqrt{u^2+1}).\qquad \cosh^{-1}u=\log(u+\sqrt{u^2-1}).$$

$$\tanh^{-1}u=\frac{1}{2}\log\frac{1+u}{1-u}.\qquad \coth^{-1}u=\frac{1}{2}\log\frac{u+1}{u-1}.$$

$$\mathrm{cosech}^{-1}u=\log\frac{1+\sqrt{1+u^2}}{u}.\qquad \mathrm{sech}^{-1}u=\log\frac{1+\sqrt{1-u^2}}{u}.$$

CHAPTER VIII

FORMULAS FOR DIFFERENTIATION OF TRIGONOMETRIC AND ANTI-TRIGONOMETRIC FUNCTIONS

79. The following formulas with those of Art. 52 will enable us to find the derivative of any trigonometric or anti-trigonometric function.

$$\text{XIII.} \qquad \frac{d \sin u}{dx} = \cos u \frac{du}{dx}.$$

$$\text{XIV.} \qquad \frac{d \cos u}{dx} = - \sin u \frac{du}{dx}.$$

$$\text{XV.} \qquad \frac{d \tan u}{dx} = \sec^2 u \frac{du}{dx}.$$

$$\text{XVI.} \qquad \frac{d \cot u}{dx} = - \operatorname{cosec}^2 u \frac{du}{dx}.$$

$$\text{XVII.} \qquad \frac{d \sec u}{dx} = \sec u \tan u \frac{du}{dx}.$$

$$\text{XVIII.} \qquad \frac{d \operatorname{cosec} u}{dx} = - \operatorname{cosec} u \cot u \frac{du}{dx}.$$

$$\text{XIX.} \qquad \frac{d \operatorname{vers} u}{dx} = \sin u \frac{du}{dx}.$$

$$\text{XX.} \qquad \frac{d \sin^{-1} u}{dx} = \frac{\frac{du}{dx}}{\sqrt{1 - u^2}}.$$

$$\text{XXI.} \qquad \frac{d \cos^{-1} u}{dx} = - \frac{\frac{du}{dx}}{\sqrt{1 - u^2}}.$$

$$\text{XXII.} \qquad \frac{d \tan^{-1} u}{dx} = \frac{\frac{du}{dx}}{1 + u^2}.$$

$$\text{XXIII.}\quad \frac{d\cot^{-1}u}{dx} = -\frac{\dfrac{du}{dx}}{1+u^2}.$$

$$\text{XXIV.}\quad \frac{d\sec^{-1}u}{dx} = \frac{\dfrac{du}{dx}}{u\sqrt{u^2-1}}.$$

$$\text{XXV.}\quad \frac{d\operatorname{cosec}^{-1}u}{dx} = -\frac{\dfrac{du}{dx}}{u\sqrt{u^2-1}}.$$

$$\text{XXVI.}\quad \frac{d\operatorname{vers}^{-1}u}{dx} = \frac{\dfrac{du}{dx}}{\sqrt{2u-u^2}}.$$

80. Derivation of some of the formulas.

Proof of XIII.

Let $\qquad y = \sin u.$

Let $x = x_0$. $\qquad \therefore\ y_0 = \sin u_0.$

Let $x = x_0 + \Delta x$. $\qquad \therefore\ y_0 + \Delta y = \sin(u_0 + \Delta u).$

Subtract. $\qquad \therefore\ \Delta y = \sin(u_0 + \Delta u) - \sin u_0.$

In the trigonometric formula,

$$\sin A - \sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B),$$

let $\qquad A = u_0 + \Delta u$, and $B = u_0$.

$$\therefore\ \sin(u_0+\Delta u) - \sin u_0 = 2\cos\tfrac{1}{2}(2u_0+\Delta u)\sin\tfrac{1}{2}\Delta u.$$

$$\therefore\ \Delta y = 2\cos\tfrac{1}{2}(2u_0+\Delta u)\sin\tfrac{1}{2}\Delta u$$

$$= 2\cos\left(u_0+\frac{\Delta u}{2}\right)\sin\tfrac{1}{2}\Delta u.$$

Divide by Δx. $\qquad \therefore\ \dfrac{\Delta y}{\Delta x} = 2\cos\left(u_0+\dfrac{\Delta u}{2}\right)\dfrac{\sin\frac{1}{2}\Delta u}{\Delta x}$

$$= 2\cos\left(u_0+\frac{\Delta u}{2}\right)\frac{\sin\frac{1}{2}\Delta u}{\Delta u}\cdot\frac{\Delta u}{\Delta x}$$

$$= \cos\left(u_0+\frac{\Delta u}{2}\right)\frac{\sin\frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}\cdot\frac{\Delta u}{\Delta x}.$$

Pass to limits.

$$\therefore \left.\frac{dy}{dx}\right|_{x=x_0} = \underset{\Delta x \doteq 0}{\text{limit}}\left[\cos\left(u_0 + \frac{\Delta u}{2}\right)\right] \cdot \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\sin \frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}\right] \cdot \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta u}{\Delta x}\right]$$

$$= \cos u_0 \left.\frac{du}{dx}\right|_{x=x_0} \cdot \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\sin \frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}\right].$$

It remains to find $\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\sin \frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}\right]$.

As $\Delta x \doteq 0$, $\Delta u \doteq 0$. $\therefore \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\sin \frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}\right] = \underset{\Delta u \doteq 0}{\text{limit}}\left[\frac{\sin \frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}\right]$.

Let $\frac{1}{2}\Delta u = \theta$. $\therefore \underset{\Delta u \doteq 0}{\text{limit}}\left[\frac{\sin \frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}\right] = \underset{\theta \doteq 0}{\text{limit}}\left[\frac{\sin \theta}{\theta}\right]$.

With O (Fig. 28) as center and any distance r as radius, describe a circle. Take any angle θ expressed in circular measure. From A draw a perpendicular to OA to meet OB produced in D.

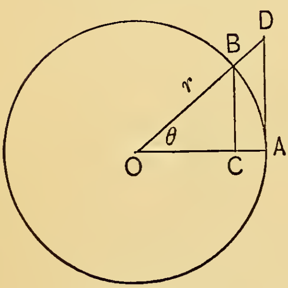


Fig. 28.

Arc $AB = r\theta$, $AD = r \tan \theta$, and $BC = r \sin \theta$.

Now by geometry,

$$CB < \text{arc } AB < AD.$$

$$\therefore r \sin \theta < r\theta < r \tan \theta.$$

Divide by r. $\therefore \sin \theta < \theta < \tan \theta$.

Divide by $\sin \theta$. $\therefore 1 < \frac{\theta}{\sin \theta} < \frac{\tan \theta}{\sin \theta}$.

$$\therefore 1 < \frac{\theta}{\sin \theta} < \sec \theta.$$

Now as $\theta \doteq 0$, $\sec \theta \doteq 1$. $\therefore \frac{\theta}{\sin \theta}$ is always greater than 1 and less than something that approaches 1 as its limit.

Therefore $\underset{\theta \doteq 0}{\text{limit}} \left[\dfrac{\sin\theta}{\theta}\right] = 1.$ $\therefore$ $\underset{\Delta x \doteq 0}{\text{limit}} \left[\dfrac{\sin\frac{1}{2}\Delta u}{\frac{1}{2}\Delta u}\right] = 1.$

$$\therefore \frac{d\sin u}{dx} = \cos u \frac{du}{dx}.$$

Proof of XIV.

Let $y = \cos u.$

$$\therefore y = \sin\left(\frac{\pi}{2} - u\right).$$

$$\therefore \frac{dy}{dx} = \cos\left(\frac{\pi}{2} - u\right) \frac{d\left(\frac{\pi}{2} - u\right)}{dx}, \text{ by Formula XIII,}$$

$$= -\cos\left(\frac{\pi}{2} - u\right) \frac{du}{dx}.$$

$$\therefore \frac{d\cos u}{dx} = -\sin u \frac{du}{dx}.$$

Proof of XV.

Let $y = \tan u.$

$$\therefore y = \frac{\sin u}{\cos u}.$$

$$\therefore \frac{dy}{dx} = \frac{\cos u \dfrac{d\sin u}{dx} - \sin u \dfrac{d\cos u}{dx}}{\cos^2 u}, \text{ by Formula VI,}$$

$$= \frac{\cos^2 u + \sin^2 u}{\cos^2 u} \frac{du}{dx}$$

$$= \frac{1}{\cos^2 u} \frac{du}{dx} = \sec^2 u \frac{du}{dx}.$$

$$\therefore \frac{d\tan u}{dx} = \sec^2 u \frac{du}{dx}.$$

Proof of XX.

Let $y = \sin^{-1} u.$

$$\therefore \sin y = u.$$

Differentiate each member of the equation with respect to x.

$$\therefore\ \cos y \frac{dy}{dx} = \frac{du}{dx}.$$

$$\therefore\ \frac{dy}{dx} = \frac{\frac{du}{dx}}{\cos u} = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

$$\therefore\ \frac{d \sin^{-1} u}{dx} = \frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

The proofs of the other formulas are left as exercises to the student.

EXERCISES

1. Prove the following:

$$\frac{d \cot u}{dx} = -\operatorname{cosec}^2 u \frac{du}{dx}. \qquad \frac{d \cot^{-1} u}{dx} = -\frac{\frac{du}{dx}}{1+u^2}.$$

$$\frac{d \operatorname{cosec} u}{dx} = -\operatorname{cosec} u \cot u \frac{du}{dx}. \qquad \frac{d \operatorname{cosec}^{-1} u}{dx} = -\frac{\frac{du}{dx}}{u\sqrt{u^2-1}}.$$

$$\frac{d \cos^{-1} u}{dx} = -\frac{\frac{du}{dx}}{\sqrt{1-u^2}}.$$

2. Prove the following:

$$\frac{d \sec u}{dx} = \sec u \tan u \frac{du}{dx}. \qquad \frac{d \sec^{-1} u}{dx} = \frac{\frac{du}{dx}}{u\sqrt{u^2-1}}.$$

$$\frac{d \operatorname{vers} u}{dx} = \sin u \frac{du}{dx}. \qquad \frac{d \operatorname{vers}^{-1} u}{dx} = \frac{\frac{du}{dx}}{\sqrt{2u-u^2}}.$$

$$\frac{d \tan^{-1} u}{dx} = \frac{\frac{du}{dx}}{1+u^2}.$$

Find $\frac{dy}{dx}$ in each of the following cases:

3. $y = \sin 3x \cos 2x$. *Ans.* $3\cos 2x \cos 3x - 2\sin 2x \sin 3x$.

4. $y = \sin^2 x \cos 2x$. *Ans.* $(1 - 4\sin^2 x)\sin 2x$.

5. $y = \sin^2 x \cos^2 x$. *Ans.* $\frac{1}{2}\sin 4x$.

6. $y = \tan^2 5x$. *Ans.* $10\tan^2 5x \sec^2 5x$.

7. $y = \log \sec^2 x$. *Ans.* $2\tan x$.

8. $y = \log \tan^2 3x$. *Ans.* $6\sec^2 3x \cot 3x$.

9. $y = \log \cos 4x$. *Ans.* $-4\tan 4x$.

10. $y = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$. *Ans.* $\sec x$.

11. $y = e^{ax}\log \sin nx$. *Ans.* $e^{ax}[n\cot nx + a\log \sin nx]$.

12. $y = e^{ax}\sin^{-1} x$. *Ans.* $e^{ax}\left[\frac{1}{\sqrt{1-x^2}} + a\sin^{-1} x\right]$.

13. $y = \tan^{-1}\frac{(e^x - e^{-x})}{2}$. *Ans.* $\frac{2}{e^x + e^{-x}}$.

14. $y = \cos^{-1}\frac{e^x - e^{-x}}{e^x + e^{-x}}$. *Ans.* $\frac{-2}{e^x + e^{-x}}$.

15. $y = \sec^{-1}\frac{2x^2}{1+x^2}$. *Ans.* $\frac{2}{x\sqrt{3x^4 - 2x^2 - 1}}$.

16. $y = \sin^{-1}\sqrt{\sin x}$. *Ans.* $\frac{1}{2}\sqrt{1 + \operatorname{cosec} x}$.

17. $y = \tan^{-1}\frac{3a^2x - x^3}{a^3 - 3ax^2}$. *Ans.* $\frac{3a}{a^2 + x^2}$.

18. $y = \sec x + \log \tan\frac{x}{2}$. *Ans.* $\sec^2 x \operatorname{cosec} x$.

19. $y = \frac{2\sin^{-1} x}{\sqrt{1-x^2}} + \log\frac{1-x}{1+x}$. *Ans.* $\frac{2x\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}}$.

20. $y = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$. *Ans.* $\sqrt{a^2 - x^2}$.

21. $y = \tan^{-1}\frac{2x - a}{a\sqrt{3}} + \tan^{-1}\frac{2a - x}{x\sqrt{3}}$. *Ans.* 0.

22. $y = \log \frac{1+x}{1-x} + \frac{1}{2} \log \frac{1+x+x^2}{1-x+x^2} + \sqrt{3} \tan^{-1} \frac{x\sqrt{3}}{1-x^2}$.

Ans. $\frac{6}{1-x^6}$.

23. In the curve $y = \log \sec x$, find the abscissas of the points at which x and y change at the same rate. *Ans.* $(4n+1)\frac{\pi}{4}$.

24. A vertical wheel of radius 8 feet makes 40 revolutions per minute in the positive direction of rotation, about a fixed axis. Find, in feet per minute, the horizontal and vertical speeds of a point on the circumference (*a*) 30° from the horizontal; (*b*) at the highest point of the circumference; (*c*) 180° from the horizontal; (*d*) at the lowest point of the circumference.

Ans. Horiz. vel.: (*a*) -320π; (*b*) -640π; (*c*) 0; (*d*) 640π.
Vert. vel.: (*a*) $320\sqrt{3}\pi$; (*b*) 0; (*c*) -640π; (*d*) 0.

25. A vertical wheel of radius 8 feet is moving about a fixed axis. If the horizontal and vertical speeds of a point on the circumference, at a given instant, are 20 feet per second and 10 feet per second, respectively, find the angle through which the wheel turns in 1 second. *Ans.* 160.1°.

26. A carriage wheel rolls along a level road, the axle moving at the rate of 8 miles per hour. Find, in feet per second, the horizontal and vertical speeds of the point of the circumference originally on the ground when the wheel has rolled through an angle of (*a*) 30°; (*b*) 60°; (*c*) 90°; (*d*) 180°; (*e*) 270°; (*f*) 360°.

Suggestion. First derive the equations of the cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

Ans. Horiz. vel. (*a*) 1.57; (*b*) 5.87; (*c*) 11.73; (*d*) 23.47; (*e*) 11.73; (*f*) 0.

Vert. vel. (*a*) 5.87; (*b*) 10.16; (*c*) 11.73; (*d*) 0; (*e*) −11.73; (*f*) 0.

27. The crank and connecting rod of a steam engine are 2 and 6 feet in length respectively. If the crank revolves uniformly at the rate of 3 turns per second, what is the rate, in feet per second, at which the piston is moving when the angle which the crank makes with the horizontal is (*a*) 0°; (*b*) 45°; (*c*) 90°; (*d*) 135°; (*e*) 180°; (*f*) 270°.

Ans. (*a*) 0; (*b*) $10.54\,\pi$; (*c*) $12\,\pi$; (*d*) $6.43\,\pi$; (*e*) 0; (*f*) $-12\,\pi$.

Determine the maximum and minimum points and points of inflexion of the following curves. Plot the curves.

28. $y = \sin x$.

Ans. Max. $(4n+1)\frac{\pi}{2}\cdot$ Min. $(4n+3)\frac{\pi}{2}\cdot$ Pts. of inflex. $n\pi$.

29. $y = \cos x$.

Ans. Max. $2n\pi$. Min. $(2n+1)\pi$. Pts. of inflex. $(2n+1)\frac{\pi}{2}\cdot$

30. $y = \sin 2x + \cos 2x$.

Ans. Max. $(8n+1)\frac{\pi}{8}\cdot$ Min. $(8n+5)\frac{\pi}{8}\cdot$

Pts. of inflex. $(4n+3)\frac{\pi}{8}\cdot$

31. $y = \sin x - \cos x$.

Ans. Max. $(8n+3)\frac{\pi}{4}\cdot$ Min. $(8n+7)\frac{\pi}{4}\cdot$

Pts. of inflex. $(4n+1)\frac{\pi}{4}\cdot$

Determine the maximum and minimum values of r in each of the two following equations. In each case, plot the curve.

32. $r = a(\sin 2\theta + \cos 2\theta)$.

33. $r = a(\sin\theta - \cos\theta)$.

34. An open gutter with cross section an arc of a circle is to be bent from a piece of tin 12 inches wide. Find the width across the top when the carrying capacity of the gutter is a maximum.

Ans. $\frac{24}{\pi}$.

CHAPTER IX

ROLLE'S THEOREM. LAW OF THE MEAN. TAYLOR'S THEOREM

81. Before proceeding to a discussion of the theorems named above, we shall prove the following theorem:

In the equation $y=f(x)$, where $f(x)$ is a single valued and continuous function of x, as x increases through x_0, if $f(x)$ is increasing, $\left.\frac{dy}{dx}\right|_{x=x_0}$ is positive or zero, and if $f(x)$ is decreasing, $\left.\frac{dy}{dx}\right|_{x=x_0}$ is negative or zero.

Proof.

Let $x=x_0$. $\qquad\therefore\ y_0=f(x_0)$.

Let $x=x_0+\Delta x$. $\qquad\therefore\ y_0+\Delta y=f(x_0+\Delta x)$.

$$\therefore\ \frac{\Delta y}{\Delta x}=\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}.$$

Suppose that $f(x)$ is increasing as x increases through x_0.

Under this supposition, $f(x_0+\Delta x)>f(x_0)$, and therefore $f(x_0+\Delta x)-f(x_0)$ or Δy is positive. Now Δx is positive, since by supposition x increases. Therefore $\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta x}\right]$ or $\left.\frac{dy}{dx}\right|_{x=x_0}$ is positive or zero.

Suppose that $f(x)$ is decreasing as x increases through x_0.

Under this supposition, $f(x_0+\Delta x)<f(x_0)$, or Δy is negative. Therefore $\frac{\Delta y}{\Delta x}$ is negative. Therefore $\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta y}{\Delta x}\right]$ or $\left.\frac{dy}{dx}\right|_{x=x_0}$ is negative or zero.

82. If between two values of x, namely $x=a$ and $x=b$, a function, $f(x)$, is single valued and continuous and has a continuous derivative, and if $f(a)=f(b)=0$, then there is at least one value x_0 of x between a and b at which $\frac{dy}{dx}$ is zero.

This is known as **Rolle's Theorem.** The geometrical statement is as follows:

If the curve whose equation is $y=f(x)$, where $f(x)$ is single valued and continuous for all values of x between two values a and b, crosses the x-axis when $x=a$ and $x=b$ and does not have a cusp for any value of x between a and b, the tangent line to the curve will be parallel to the x-axis for at least one value of x between a and b.

Proof of Rolle's Theorem.

Since $f(x)$ is zero when $x=a$ and $x=b$, and is continuous, it must have:

Case I been zero for all values of x between a and b, or

Case II increased from zero and finally decreased back to zero, or

Case III decreased from zero and finally increased back to zero.

Case I. If $f(x)$ is zero for all values of x between a and b, its derivative, $\frac{dy}{dx}$, is zero, not only for one value, but for all values of x between a and b.

Case II. If $f(x)$ increased from zero and finally decreased back to zero, it must have changed from increasing to decreasing at least once. Therefore, by Art. 81, its derivative, $\frac{dy}{dx}$, must have changed from positive to negative at least once. Now a continuous function can change in sign only by passing through the value zero. Therefore $\frac{dy}{dx}$ must have been zero for at least one value of x between a and b.

The Proof of Case III is entirely similar to that of Case II.

83. As an illustration, consider the equation

$$y = x^4 - 8x^3 + 22x^2 - 24x.$$

The function y is zero when $x = 0$ and $x = 4$. Then, by Rolle's Theorem, the derivative must be zero for at least one value of x between 0 and 4. That such is the case can readily be seen.

$$\frac{dy}{dx} = 4x^3 - 24x^2 + 44x - 24$$
$$= 4(x-1)(x-2)(x-3);$$

and an examination of this function shows that it vanishes three times between 0 and 4, namely, at 1, 2, and 3.

EXERCISES

1. Give a geometrical proof of Rolle's Theorem.

In each of the following equations, show that $\frac{dy}{dx}$ is zero for at least one value of x between each two for which y is zero. In each case, plot the curve.

2. $y = x^3 + 6x^2 + 11x + 6.$

3. $y = x^3 - 12x + 11.$

4. $y = x^4 - 3x^3 + x^2 + 3x - 2.$

5. $y = x^3 - x^2 - 16x + 16.$

84. If between two values of x, namely $x = a$ and $x = b$, a function, $f(x)$, is single valued and continuous and has a continuous derivative, then there must be at least one value x_1 of x between a and b at which

$$\frac{f(b) - f(a)}{b - a} = f'(x_1).$$

This is known as the **Law of the Mean**. It is readily seen that Rolle's Theorem is but a special case of it, because if $f(a) = f(b) = 0$, $\frac{f(b) - f(a)}{b - a} = f'(x_1)$, $a < x_1 < b$, becomes $0 = f'(x_1)$, $a < x_1 < b$, which is Rolle's Theorem.

85. The Law of the Mean has a simple geometrical interpretation. Plot the curve $y=f(x)$ (see Fig. 29). The left-hand member, $\frac{f(b)-f(a)}{b-a}$, is the slope of the chord AB. The right-hand member, $f'(x_1)$, $a < x_1 < b$, is the slope of the tangent line to the curve at a point on the curve for which x has a value between a and b. The Law of the Mean stated geometrically is therefore:

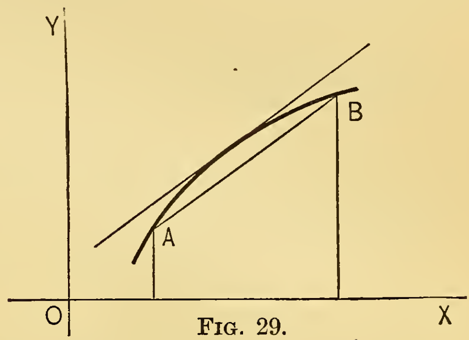


Fig. 29.

In the curve $y=f(x)$ in which $f(x)$ is single valued and continuous and has a continuous derivative for all values of x between a and b, the slope of the chord joining the points for which $x=a$ and $x=b$ is equal to the slope of the tangent line at some point for which x has a value between a and b.

86. To establish the Law of the Mean, we proceed as follows:

From $f(x)$ we build up a function, $F(x)$, which is zero when $x=a$ and $x=b$. Then, by applying Rolle's Theorem to $F(x)$, we are able to obtain the desired result.

Proof of the Law of the Mean.

Let
$$\frac{f(b)-f(a)}{b-a}=K. \tag{1}$$

Since a and b are constants, K is constant.

Clear of fractions and transpose.

$$\therefore\ f(b)-f(a)-(b-a)K=0. \tag{2}$$

From $f(x)$ build up the function

$$F(x)\equiv f(x)-f(a)-(x-a)K,$$

where K has the value given it in (1).

Then $\qquad F(a) \equiv f(a) - f(a) - (a - a) K = 0.$

Also, $\qquad F(b) \equiv f(b) - f(a) - (b - a) K$

$$= 0 \text{ because of (2).}$$

Differentiate $F(x)$, remembering that $f(a)$ and K are constants.

$$\therefore \; F'(x) \equiv f'(x) - K.$$

Since $F(a) = F(b) = 0$, $\therefore$ $F'(x_1) \equiv f'(x_1) - K = 0$, $a < x_1 < b$, by Rolle's Theorem.

$$\therefore \; K = f'(x_1), \quad a < x_1 < b.$$

Substitute in (1).

$$\therefore \; \frac{f(b) - f(a)}{b - a} = f'(x_1), \quad a < x_1 < b.$$

This equation is frequently written in the form

$$f(b) = f(a) + (b - a) f'(x_1), \quad a < x_1 < b.$$

87. If between two values of x, namely $x = a$ and $x = b$, a function, $f(x)$, is single valued and continuous and has a first and second derivative each of which is continuous, then

$$f(b) = f(a) + (b - a) f'(a) + \frac{(b - a)^2}{\lfloor 2} f''(x_1), \quad a < x_1 < b.$$

Proof.

Let $$\frac{f(b) - f(a) - (b - a) f'(a)}{\dfrac{(b - a)^2}{\lfloor 2}} = K. \qquad (1)$$

Since a and b are constants, K is a constant.

Clear of fractions and transpose.

$$\therefore \; f(b) - f(a) - (b - a) f'(a) - \frac{(b - a)^2}{\lfloor 2} K = 0. \qquad (2)$$

From $f(x)$ build up the function

$$F(x) \equiv f(x) - f(a) - (x - a) f'(a) - \frac{(x - a)^2}{\lfloor 2} K,$$

where K has the value given it in (1).

Then $F(a) \equiv f(a) - f(a) - (a-a)f'(a) - \frac{(a-a)^2}{\lfloor 2} K = 0.$

Also, $F(b) \equiv f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{\lfloor 2} K$

$= 0$ because of (2).

Differentiate $F(x)$, remembering that $f(a)$, $f'(a)$, and K are constants.

$$\therefore\ F'(x) \equiv f'(x) - f'(a) - (x-a)K.$$

Since $F(a) = F(b) = 0$,

$$\therefore\ F'(h_1) = f'(h_1) - f'(a) - (h_1 - a)K = 0,\quad a < h_1 < b, \qquad (3)$$

by Rolle's Theorem.

Differentiate $F'(x)$.

$$\therefore\ F''(x) \equiv f''(x) - K.$$

Now $F'(a) = f'(a) - f'(a) - (a-a)K = 0$. Also, $F'(h_1) = 0$ because of (3).

$$\therefore\ F''(h_2) \equiv f''(h_2) - K = 0,\quad a < h_2 < h_1,$$

by Rolle's Theorem.

$$\therefore\ K = f''(h_2),\quad a < h_2 < h_1.$$

Since $h_2 < h_1 < b$, $\therefore\ h_2 < b$.

$$\therefore\ K = f''(h_2),\quad a < h_2 < b.$$

Substitute in (1).

$$\therefore\ \frac{f(b) - f(a) - (b-a)f'(a)}{\frac{(b-a)^2}{\lfloor 2}} = f''(h_2),\quad a < h_2 < b.$$

Call $h_2 = x_1$. Clear of fractions and transpose.

$$\therefore\ f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(x_1),\quad a < x_1 < b.$$

EXERCISES

1. Find, by the Law of the Mean, a point on the curve $y = x^3$ at which the tangent line to the curve is parallel to the line joining the points (2, 8) and (5, 125). *Ans.* $(\sqrt{13},\ 13\sqrt{13})$.

2. Using the Law of the Mean, find the equation of the tangent line to the curve $y^2 = 4x$, which is parallel to the line joining the points (0, 0) and (4, 4). *Ans.* $x - y + 1 = 0$.

3. Using the Law of the Mean, find the equation of the tangent line to the curve $y^2 = 8x$, which is parallel to the line $2x - 3y + 8 = 0$. *Ans.* $2x - 3y + 9 = 0$.

4. Show that $F(x)$ in Art. 86 represents the length QP where P and Q are points on the curve $y = f(x)$ and the chord AB respectively having the same abscissa x, $a < x < b$.

5. Express $3x^2 - 2x + 5$ in powers of $x - 2$.
Ans. $3(x-2)^2 + 10(x-2) + 13$.

6. If between two values of x, namely $x = a$ and $x = b$, a function, $f(x)$, is single valued and continuous, and has a first, second, and third derivative, each of which is continuous, then

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{\underline{|2}} f''(a) + \frac{(b-a)^3}{\underline{|3}} f'''(x_1),\ a < x_1 < b.$$

Suggestion. Let

$$\frac{f(b) - f(a) - (b-a)f'(a) - \dfrac{(b-a)^2}{\underline{|2}} f''(a)}{\dfrac{(b-a)^3}{\underline{|3}}} = K, \qquad \text{and}$$

$$F(x) \equiv f(x) - f(a) - (x-a)f'(a) - \frac{(x-a)^2}{\underline{|2}} f''(a) - \frac{(x-a)^3}{\underline{|3}} K.$$

Using the results of Exercise 6, express:

7. $2x^3 - 3x^2 + 4x - 5$ in powers of $x + 1$.
Ans. $2(x+1)^3 - 9(x+1)^2 + 16(x+1) - 14$.

8. $3x^3 - 2x^2 - 4x + 2$ in powers of $x - 1$.
Ans. $3(x-1)^3 + 7(x-1)^2 + (x-1) - 1$.

88. If between two values of x, namely $x=a$ and $x=b$, a function, $f(x)$, is single valued and continuous, and has $n+1$ derivatives all of which are continuous, then

$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{\lfloor 2}f''(a)+\frac{(b-a)^3}{\lfloor 3}f'''(a)+\cdots$$
$$+\frac{(b-a)^n}{\lfloor n}f^{(n)}a+\frac{(b-a)^{n+1}}{\lfloor n+1}f^{(n+1)}(x_1),\quad a<x_1<b.$$

This is known as **Taylor's Theorem.** It is readily seen to be but a more general form of the Law of the Mean, and the proof is exactly similar to that given for that theorem and for the one of Art. 87.

Proof. Let

$$\frac{f(b)-f(a)-(b-a)f'(a)-\dfrac{(b-a)^2}{\lfloor 2}f''(a)-\dfrac{(b-a)^3}{\lfloor 3}f'''(a)-\cdots-\dfrac{(b-a)^n}{\lfloor n}f^{(n)}(a)}{\dfrac{(b-a)^{n+1}}{\lfloor n+1}}=K. \quad (1)$$

Since a and b are constants, K is a constant.

Clear of fractions and transpose.

$$\therefore\ f(b)-f(a)-(b-a)f'(a)-\frac{(b-a)^2}{\lfloor 2}f''(a)-\frac{(b-a)^3}{\lfloor 3}f'''(a)-\cdots$$
$$-\frac{(b-a)^n}{\lfloor n}f^{(n)}(a)-\frac{(b-a)^{n+1}}{\lfloor n+1}K=0. \quad (2)$$

From $f(x)$ form the function

$$F(x)\equiv f(x)-f(a)-(x-a)f'(a)-\frac{(x-a)^2}{\lfloor 2}f''(a)-\frac{(x-a)^3}{\lfloor 3}f'''(a)$$
$$-\cdots-\frac{(x-a)^n}{\lfloor n}f^{(n)}(a)-\frac{(x-a)^{n+1}}{\lfloor n+1}K, \quad (3)$$

where K has the value given it in (1).

$$F(a) \equiv f(a) - f(a) - (a-a)f'(a) - \frac{(a-a)^2}{|\underline{2}}f''(a) - \frac{(a-a)^3}{|\underline{3}}f'''(a)$$

$$- \cdots - \frac{(a-a)^n}{|\underline{n}}f^{(n)}(a) - \frac{(a-a)^{n+1}}{|\underline{n+1}}K = 0.$$

$$F(b) = f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{|\underline{2}}f''(a) - \frac{(b-a)^3}{|\underline{3}}f'''(a)$$

$$- \cdots - \frac{(b-a)^n}{|\underline{n}}f^{(n)}(a) - \frac{(b-a)^{n+1}}{|\underline{n+1}}K = 0 \text{ because of (2).}$$

Differentiate (3).

$$\therefore\ F'(x) \equiv f'(x) - f'(a) - (x-a)f''(a) - \frac{(x-a)^2}{|\underline{2}}f'''(a) - \cdots$$

$$- \frac{(x-a)^{n-1}}{|\underline{n-1}}f^{(n)}(a) - \frac{(x-a)^n}{|\underline{n}}K. \tag{4}$$

Since $F(a) = F(b) = 0$, $\therefore\ F'(h_1) = 0$, where $a < h_1 < b$, by Rolle's Theorem.

Also, $F'(a) \equiv f'(a) - f'(a) - (a-a)f''(a) - \dfrac{(a-a)^2}{|\underline{2}}f'''(a)$

$$- \cdots - \frac{(a-a)^{n-1}}{|\underline{n-1}}f^{(n)}(a) - \frac{(a-a)^n}{|\underline{n}}K = 0.$$

Differentiate (4).

$$\therefore\ F''(x) \equiv f''(x) - f''(a) - (x-a)f'''(x) - \cdots$$

$$- \frac{(x-a)^{n-2}}{|\underline{n-2}}f^{(n)}(a) - \frac{(x-a)^{n-1}}{|\underline{n-1}}K.$$

Since $F'(a) = F'(h_1) = 0$, $\therefore\ F''(h_2) = 0$, where $a < h_2 < h_1$, by Rolle's Theorem. Also, $F''(a) = 0$. Therefore, by Rolle's Theorem, $F'''(h_3) = 0$, where $a < h_3 < h_2$. And thus we can go on. After $n+1$ differentiations there results $F^{(n+1)}(x) \equiv f^{(n+1)}(x) - K$,

which is zero for some value h_n such that $a < h_n < h_{n-1}$, where h_{n-1} is the value of x for which $F^{(n)}(x) = 0$.

Since $a < h_n < h_{n-1} < \cdots < h_3 < h_2 < h_1 < b$, $\therefore\ a < h_n < b$.

$$\therefore\ K = f^{(n+1)}(h_n),\ \ a < h_n < b.$$

Substitute in (1).

$$\therefore\ \frac{f(b) - f(a) - (b-a)f'(a) - \dfrac{(b-a)^2}{\lfloor 2}f''(a) - \dfrac{(b-a)^3}{\lfloor 3}f'''(a) - \cdots - \dfrac{(b-a)^n}{\lfloor n}f^{(n)}(a)}{\dfrac{(b-a)^{n+1}}{\lfloor n+1}}$$

$$= f^{(n+1)}(h_n),\ \ a < h_n < b.$$

Call $h_n = x_1$. Clear of fractions and transpose.

$$\therefore\ f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{\lfloor 2}f''(a) + \frac{(b-a)^3}{\lfloor 3}f'''(a) + \cdots$$

$$+ \frac{(b-a)^n}{\lfloor n}f^{(n)}(a) + \frac{(b-a)^{n+1}}{\lfloor n+1}f^{(n+1)}(x_1),\ \ a < x_1 < b. \qquad (5)$$

89. In Equation (5) of the preceding article, we have an expression which holds true for all values of x from a up to a value b at which $f(x)$ or one of its derivatives ceases to be finite or continuous. If then we substitute x for b, we have an expression which holds true from a up to any value x at which $f(x)$ or one of its derivatives ceases to be finite or continuous. Substitute x for b.

$$\therefore\ f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2}f''(a) + \frac{(x-a)^3}{\lfloor 3}f'''(a) + \cdots$$

$$+ \frac{(x-a)^n}{\lfloor n}f^{(n)}(a) + \frac{(x-a)^{n+1}}{\lfloor n+1}f^{(n+1)}(x_1),\ a < x_1 < x.$$

This is another form of Taylor's Theorem.

90. Definitions. If $f(x)$ is expressed as

$$f(a)+(x-a)f'(a)+\frac{(x-a)^2}{\underline{|2}}f''(a)+\frac{(x-a)^3}{\underline{|3}}f'''(a)+\cdots$$
$$+\frac{(x-a)^n}{\underline{|n}}f^{(n)}(a)+\frac{(x-a)^{n+1}}{\underline{|n+1}}f^{(n+1)}(x_1),\quad a<x_1<x,$$

it is said to be developed into a power series in $x-a$ with a remainder.

The remainder is $\dfrac{(x-a)^{n+1}}{\underline{|n+1}}f^{(n+1)}(x_1),\quad a<x_1<x.$

91. If a is chosen zero, Taylor's Theorem becomes

$$f(x)=f(0)+xf'(0)+\frac{x^2}{\underline{|2}}f''(0)+\frac{x^3}{\underline{|3}}f'''(0)+\cdots$$
$$+\frac{x^n}{\underline{|n}}f^{(n)}(0)+\frac{x^{n+1}}{\underline{|n+1}}f^{(n+1)}(x_1),\quad 0<x_1<x,$$

which is a power series in x with a remainder.

This special form of Taylor's Theorem is known as **Maclauren's** or **Stirling's Theorem.**

92. Example 1. Develop $\cos x$ into a power series in $x-a$ consisting of five terms and the remainder.

$$\begin{aligned}
f(x) &\equiv \cos x, & \therefore\ f(a) &= \cos a.\\
f'(x) &\equiv -\sin x, & \therefore\ f'(a) &= -\sin a.\\
f''(x) &\equiv -\cos x, & \therefore\ f''(a) &= -\cos a.\\
f'''(x) &\equiv \sin x, & \therefore\ f'''(a) &= \sin a.\\
f^{\text{IV}}(x) &\equiv \cos x, & \therefore\ f^{\text{IV}}(a) &= \cos a.\\
f^{\text{V}}(x) &\equiv -\sin x, & \therefore\ f^{\text{V}}(a) &= -\sin a.
\end{aligned}$$

$$\therefore\ \cos x=\cos a-(x-a)\sin a-\frac{(x-a)^2}{\underline{|2}}\cos a+\frac{(x-a)^3}{\underline{|3}}\sin a$$
$$+\frac{(x-a)^4}{\underline{|4}}\cos a-\frac{(x-a)^5}{\underline{|5}}\sin x_1,\quad a<x_1<x.$$

EXAMPLE 2. Develop $\sin x$ into a power series in x consisting of n terms and the remainder.

$$\begin{aligned}
f(x) &\equiv \sin x, &\therefore f(0) &= 0.\\
f'(x) &\equiv \cos x, &\therefore f'(0) &= 1.\\
f''(x) &\equiv -\sin x, &\therefore f''(0) &= 0.\\
f'''(x) &\equiv -\cos x, &\therefore f'''(0) &= -1.\\
f^{\text{IV}}(x) &\equiv \sin x, &\therefore f^{\text{IV}}(0) &= 0.
\end{aligned}$$

The law according to which the derivatives proceed is that all derivatives of even order are $\sin x$ with the $+$ or $-$ sign attached, and all of odd order are $\cos x$ with the $+$ or $-$ sign attached. Also, that the sines are alternately $+$ and $-$, and also the cosines. Since each alternate term is zero, we must take $2n$ derivatives in order to get a series of n terms and the remainder.

$$\therefore f^{(2n-1)}(x) \equiv (-1)^{n-1}\cos x, \quad \therefore f^{(2n-1)}(x) = (-1)^{n-1}\cdot 1.$$

$$f^{(2n)}(x) \equiv (-1)^n \sin x, \quad \therefore f^{(2n)}(x_1) = (-1)^n \sin x_1.$$

$$\therefore \sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{\lfloor 2n-1} + (-1)^n\frac{x^{2n}}{\lfloor 2n}\sin x_1,$$

$$0 < x_1 < x.$$

EXAMPLE 3. Develop $\log(1+x)$ into a power series in x consisting of n terms and the remainder.

$$\begin{aligned}
f(x) &\equiv \log(1+x), &\therefore f(0) &= 0.\\
f'(x) &\equiv \frac{1}{1+x}, &\therefore f'(0) &= 1.\\
f''(x) &\equiv -\frac{1}{(1+x)^2}, &\therefore f''(0) &= -1.\\
f'''(x) &\equiv \frac{1\cdot 2}{(1+x)^3}, &\therefore f'''(0) &= 1\cdot 2.\\
f^{\text{IV}}(x) &\equiv -\frac{1\cdot 2\cdot 3}{(1+x)^4}, &\therefore f^{\text{IV}}(0) &= -1\cdot 2\cdot 3.
\end{aligned}$$

Since $f(x)=0$, it is necessary to take $n+1$ derivatives in order to get a series of n terms and the remainder.

$$f^{(n)}(x) \equiv (-1)^{n-1}\frac{\lfloor n-1}{(1+x)^n}, \quad \therefore f^{(n)}(0)=(-1)^{n-1}\lfloor n-1.$$

$$f^{(n+1)}(x) \equiv (-1)^{n}\frac{\lfloor n}{(1+x)^{n+1}}, \quad \therefore f^{(n+1)}(x_1)=(-1)^{n}\frac{\lfloor n}{(1+x_1)^{n+1}}.$$

$$\therefore \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+(-1)^{n-1}\frac{x^n}{n}$$
$$+(-1)^n\frac{x^{n+1}}{n+1}\cdot\frac{1}{(1+x_1)^{n+1}}, \quad 0<x_1<x.$$

EXERCISES

1. Express $x^4-3x^3+2x^2-x+2$ in powers of $x+2$.

Ans. $(x+2)^4-11(x+2)^3+44(x+2)^2-77(x+2)+52$.

2. Express $x^5+3x^4-4x^3+2x^2-x+2$ in powers of $x-1$.

Ans. $(x-1)^5+8(x-1)^4+18(x-1)^3+18(x-1)^2+8(x-1)+3$.

3. Reduce the roots of the equation $x^5-2x^3+3x^2-x+1=0$ by 2, first by Horner's method, second by Taylor's Theorem.

Ans. The equation with reduced roots is $x^5+10x^4+38x^3+71x^2+67x+27=0$.

4. Develop $\cos x$ into a power series in x consisting of n terms and the remainder.

Ans. $$\cos x = 1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\frac{x^6}{\lfloor 6}+\cdots+(-1)^{n-1}\frac{x^{2n-2}}{\lfloor 2n-2}$$
$$+(-1)^n\frac{x^{2n-1}}{\lfloor 2n-1}\sin x_1, \quad 0<x_1<x.$$

5. Develop $\sin x$ into a power series in $x-a$ consisting of five terms and the remainder.

Ans. $$\sin x=\sin a+(x-a)\cos a-\frac{(x-a)^2}{\lfloor 2}\sin a-\frac{(x-a)^3}{\lfloor 3}\cos a$$
$$+\frac{(x-a)^4}{\lfloor 4}\sin a+\frac{(x-a)^5}{\lfloor 5}\cos x_1, \quad a<x_1<x.$$

6. Develop e^x into a power series in x consisting of n terms and the remainder.

$$Ans.\ e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots + \frac{x^{n-1}}{\underline{|n-1}} + \frac{x^n}{\underline{|n}} e^{x_1},\ 0 < x_1 < x.$$

7. Find the first three terms of the development of $\log_{10} \cos x$ as a power series in x.

Ans. The first three terms of the development are

$$-M\left[\frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}\right],\ \text{where } M = \log_{10} e = 0.434294 \cdots.$$

8. Develop e^x into a power series in $x + a$ consisting of n terms and the remainder.

$$Ans.\ e^x = e^{-a}\left[1 + (x+a) + \frac{(x+a)^2}{\underline{|2}} + \frac{(x+a)^3}{\underline{|3}} + \cdots + \frac{(x+a)^{n-1}}{\underline{|n-1}} + \frac{(x+a)^n}{\underline{|n}} e^{x_1}\right],\ -a < x_1 < x.$$

9. Develop e^{3x} into a power series in x consisting of n terms and the remainder.

$$Ans:\ e^{3x} = 1 + 3x + \frac{3^2x^2}{\underline{|2}} + \frac{3^3x^3}{\underline{|3}} + \cdots + \frac{3^{n-1}x^{n-1}}{\underline{|n-1}} + \frac{3^nx^n}{\underline{|n}} e^{3x_1},$$

$$0 < x_1 < x.$$

10. Find the value of $\dfrac{\log_{10} x}{x - 1}$ when $x = 1$.

Suggestion. Develop $\log_{10} x$ into a power series in $x-1$ consisting of one term and the remainder. *Ans.* $0.434 \cdots$.

11. Find the value of $\dfrac{x \sin x}{x - 2 \sin x}$ when $x = 0$. *Ans.* 0.

12. Find the value of $\dfrac{e^x + e^{-x} - 2}{x \sin x}$ en $x = 0$. *Ans.* 1.

13. Find the value of $\dfrac{\log \cos 2x}{\log \cos x}$ when $x = 0$. *Ans.* 4.

14. Find the value of $(\cos 2x)^{\frac{1}{x^2}}$ when $x = 0$.

Suggestion. Let $u = (\cos 2x)^{\frac{1}{x^2}}$. $\therefore \log u = \dfrac{\log(\cos 2x)}{x^2}$.

Develop $\log(\cos 2x)$ into a power series in x consisting of two terms and the remainder.

Ans. $\frac{1}{e^2}$.

15. Find the value of $(a^2 - x^2) \tan \frac{\pi x}{2a}$ when $x = a$.

Ans. $\frac{4a^2}{\pi}$.

16. Find the value of $\left(3 - \frac{x}{a}\right)^{\tan \frac{\pi x}{4a}}$ when $x = 2a$.

Ans. $e^{\frac{4}{\pi}}$.

93. In the development

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{\lfloor 2}f''(a) + \cdots + \frac{(x-a)^n}{\lfloor n}f^{(n)}(a)$$
$$+ \frac{(x-a)^{n+1}}{\lfloor n+1}f^{(n+1)}(x_1), \quad a < x_1 < x,$$

the sum of the first n terms and the remainder is equal to $f(x)$ for all values of x for which $f(x)$ and its first $n+1$ derivatives are continuous. Let n increase without limit. If

$$\operatorname*{limit}_{n=\infty}\left[\frac{(x-a)^{n+1}}{\lfloor n+1}f^{(n+1)}(x_1)\right] = 0,$$

the limit of the sum of the first n terms is $f(x)$. That is, the series becomes an infinite series which is convergent and converges to the value $f(x)$. Then, for those values of x for which the remainder approaches the limit zero, the function is equal to the infinite series.

From this infinite series the value of the function for particular values of x can be calculated.

For example, the development of $\sin x$ into a power series in x is

$$\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots \pm \frac{x^{2n-1}}{\lfloor 2n-1} \mp \frac{x^{2n}}{\lfloor 2n}\sin x_1, \quad 0 < x_1 < x.$$

Since $\sin x_1$ must be less in absolute value than 1, $\frac{x^{2n}}{\lfloor 2n}$ must be greater in absolute value than $\frac{x^{2n}}{\lfloor 2n}\sin x_1$. Now $\operatorname*{limit}_{n=\infty}\left[\frac{x^{2n}}{\lfloor 2n}\right]$

is evidently zero for all values of x. Therefore $\lim\limits_{n=\infty}\left[\frac{x^{2n}}{\underline{|2n}}\sin x_1\right]$ is zero for all values of x. Therefore, for all values of x, $\sin x$ is equal to the infinite series

$$x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\frac{x^7}{\underline{|7}}+\cdots.$$

To calculate the value of $\sin x$ where x has some particular value, all that is necessary is to substitute that value in the infinite series and calculate enough terms to get the result to the required number of decimal places.

EXERCISES

1. Show that $\cos x$ is equal to the infinite series

$$1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}-\frac{x^6}{\underline{|6}}+\cdots$$

for all values of x.

2. Show that $\log(1+x)$ is equal to the infinite series

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

if $|x|<1$.

CHAPTER X

BINOMIAL THEOREM

94. Develop $(1+x)^m$ into a power series in x consisting of $n+1$ terms and the remainder.

$$f(x)=(1+x)^m, \qquad \therefore \quad f(0)=1.$$

$$f'(x)=m(1+x)^{m-1}, \qquad \therefore \quad f'(0)=m.$$

$$f''(x)=m(m-1)(1+x)^{m-2}, \qquad \therefore \quad f''(0)=m(m-1).$$

$$f'''(x)=m(m-1)(m-2)(1+x)^{m-3},$$

$$\therefore \quad f'''(0)=m(m-1)(m-2).$$

. . . .

$$f^{(n)}(x)=m(m-1)(m-2)\cdots(m-n+1)(1+x)^{m-n},$$

$$\therefore \quad f^{(n)}(0)=m(m-1)(m-2)\cdots(m-n+1).$$

$$f^{(n+1)}(x)=m(m-1)(m-2)\cdots(m-n+1)(m-n)(1+x)^{m-n-1},$$

$$\therefore \quad f^{(n+1)}(x_1)=m(m-1)(m-2)\cdots(m-n+1)(m-n)(1+x_1)^{m-n-1}$$

$$\therefore \quad (1+x)^m=1+mx+\frac{m(m-1)}{\underline{|2}}x^2+\frac{m(m-1)(m-2)}{\underline{|3}}x^3+\cdots$$

$$+\frac{m(m-1)(m-2)\cdots(m-n+1)}{\underline{|n}}x^n$$

$$+\frac{m(m-1)(m-2)\cdots(m-n+1)(m-n)}{\underline{|n+1}}x^{n+1}(1+x_1)^{m-n-1}.$$

If m is a positive integer, $f^{(m)}(x)=\underline{|m}$, a constant, and hence all subsequent derivatives are zero. In this case the series

consists of a finite number, $m+1$, of terms, and $(1+x)^m$ is equal to the $m+1$ terms of the series

$$1 + mx + \frac{m(m-1)}{\lfloor\!\underline{2}}x^2 + \frac{m(m-1)(m-2)}{\lfloor\!\underline{3}}x^3 + \cdots,$$

whose $(n+1)$th term is

$$\frac{m(m-1)(m-2)\cdots(m-n+1)}{\lfloor\!\underline{n}}x^n.$$

If m is not a positive integer, the series does not consist of a finite number of terms, but becomes an infinite series as n increases without limit. The infinite series is equal to the function $(1+x)^m$ for those values of x, and only those, for which the remainder

$$\frac{m(m-1)(m-2)\cdots(m-n+1)(m-n)}{\lfloor\!\underline{n+1}}x^{n+1}(1+x_1)^{m-n-1}$$

approaches zero as a limit as n increases without limit.

It can be proved without much difficulty, although the work is too long and complicated to be given here, that this remainder approaches the limit zero for all values of x less than 1 and greater than -1. We shall assume the truth of this statement and say that, for those values of x less than 1 and greater than -1, $(1+x)^m$ is equal to the infinite series

$$1 + mx + \frac{m(m-1)}{\lfloor\!\underline{2}}x^2 + \frac{m(m-1)(m-2)}{\lfloor\!\underline{3}}x^3 + \cdots,$$

whose $(n+1)$th term is

$$\frac{m(m-1)(m-2)\cdots(m-n+1)}{\lfloor\!\underline{n}}x^n.$$

95. The above is but a special case of the development of $(a+h)^m$, where a and h are any numbers.

Suppose that it is required to develop $(a+h)^m$. If a is greater than h, divide and multiply $(a+h)^m$ by a^m. $\therefore (a+h)^m = a^m\left(1+\frac{h}{a}\right)^m$. Since a is greater than h, $\frac{h}{a}$ is less than 1. Let $\frac{h}{a}=x$. $\therefore$ x is less than 1. Therefore the development in Art. 94 holds. That is:

$$\left(1+\frac{h}{a}\right)^m = 1 + m\frac{h}{a} + \frac{m(m-1)}{\underline{|2}}\frac{h^2}{a^2} + \frac{m(m-1)(m-2)}{\underline{|3}}\frac{h^3}{a^3} + \cdots.$$

$$\therefore (a+h)^m = a^m + ma^{m-1}h + \frac{m(m-1)}{\underline{|2}}a^{m-2}h^2$$

$$+\frac{m(m-1)(m-2)}{\underline{|3}}a^{m-3}h^3 + \cdots, \tag{1}$$

the $(n+1)$th term being

$$\frac{m(m-1)(m-2)\cdots(m-n+1)}{\underline{|n}}a^{m-n}h^n.$$

If a is less than h, $(a+h)^m$ may be written as $(h+a)^m$. Divide and multiply $(h+a)^m$ by h^m.

$$\therefore (h+a)^m = h^m\left(1+\frac{a}{h}\right)^m$$

$$= h^m + mh^{m-1}a + \frac{m(m-1)}{\underline{|2}}h^{m-2}a^2$$

$$+\frac{m(m-1)(m-2)}{\underline{|3}}h^{m-3}a^3 + \cdots, \tag{2}$$

the $(n+1)$th term being

$$\frac{m(m-1)(m-2)\cdots(m-n+1)}{\underline{|n}}h^{m-n}a^n.$$

Either of the developments (1) or (2) is called the **Binomial Theorem.** It is seen to be but a special case of Taylor's Theorem.

EXERCISES

1. Find the first n terms of the development of $\tan^{-1} x$ into a power series in x.

SUGGESTION. $f(x) = \tan^{-1} x$.

$$\therefore\ f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots, \text{ if } x<1,$$

and $f''(x) = -2x + 4x^3 - 6x^5 + \cdots$, if $x < 1$, etc.

Ans. The first n terms of the development are:

$$x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1}.$$

2. Find the first four terms of the development of $\sin^{-1} x$ into a power series in x.

Ans. The first four terms are:

$$x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7}.$$

3. Find the first five terms of the development of $\cos^{-1} x$ into a power series in x.

Ans. The first five terms are:

$$\frac{\pi}{2} - x - \frac{1}{2} \cdot \frac{x^3}{3} - \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7}.$$

4. From the result of Exercise 1, calculate π to two decimal places.

CHAPTER XI

DIFFERENTIALS

96. Up to this point $\frac{dy}{dx}$ in the equation $y=f(x)$ was treated as a symbol and not as a fraction. It is, however, sometimes convenient to treat it as a fraction, but in order to do so, definitions must first be made for dx and dy.

Definitions. In a function of one variable, the **differential of the variable** is defined to be the same as the increment of the variable.

In a function of one variable, the **differential of the function** is defined to be the derivative of the function with respect to the variable multiplied by the differential or increment of the variable.

Thus, in the equation $y=f(x)$, the differential of x is defined to be the same as Δx and the differential of y to be the derivative of y with respect to x, multiplied by either the differential of x or Δx.

Differentials are denoted by writing d before the variable or function. Thus, in the equation $y=f(x)$, the differential of x is denoted by dx and the differential of y by dy or $df(x)$.

97. In the equation $y=f(x)$,

$$dy = \text{derivative} \times dx, \text{ by definition.}$$

$$\therefore \frac{dy}{dx} = \text{derivative.}$$

Therefore the derivative may be looked upon as a fraction: — the ratio of two differentials, as well as a symbol: — the limit of the ratio of two increments.

The expression $\frac{dy}{dx}$ is the ratio of two differentials. Hence the name Differential Calculus.

98. We saw that the derivative is geometrically the slope of the tangent line to the curve at a point (x, y) on the curve. Let us see what dx and dy are geometrically.

Let P, (x, y), be a point on the curve. (Fig. 30.) Let PA be the tangent line at P. Give x the increment Δx. Draw the ordinate FB and produce it to meet the tangent line at A. From P draw PC parallel to the x-axis to meet FB in C.

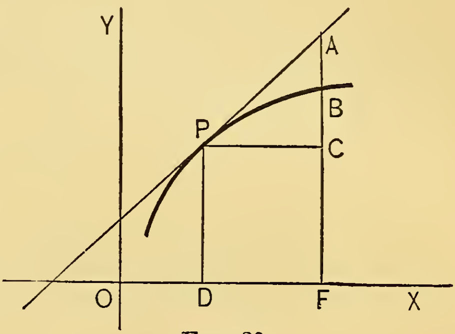


Fig. 30.

By definition, dx is the same as Δx. Therefore dx is geometrically any arbitrarily chosen length DF. By definition, dy is the derivative of y with respect to x multiplied by dx. Now the derivative of y with respect to x is $\tan CPA$. Therefore $dy = dx \tan CPA$. Also, $CA = dx \tan CPA$. Therefore $dy = CA$. Therefore dy is the distance, perpendicular to the x-axis, from C to the tangent line.

It will be remembered that Δy is the distance perpendicular to the x-axis from C to the curve.

99. Our formulas for finding derivatives can be readily converted into formulas in differentials.

Example 1. If $$y = u + v,$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

Since the derivatives are fractions, we may cancel dx.

$$\therefore\ dy = du + dv.$$

EXAMPLE 2. If $y = uv$,

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Cancel dx. $\therefore\ dy = u\,dv + v\,du.$

EXAMPLE 3. If $y = \frac{u}{v}$,

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Cancel dx. $\therefore\ dy = \frac{v\,du - u\,dv}{v^2}.$

EXERCISES

1. In the equation $y = x^3$, calculate Δy and dy when $x = 2$ and $\Delta x = 3$. Represent geometrically.

Ans. $\Delta y = 117$; $dy = 36$.

2. In the equation $y = \sin x$, calculate Δy and dy when $x = \frac{\pi}{4}$, $\Delta x = \frac{\pi}{8}\cdot$ Represent geometrically.

Ans. $\Delta y = 0.2777$; $dy = 0.278$.

In each of the following equations, find dy for any values of x and dx.

3. $y = x\log(x^2+1)$. *Ans.* $dy = \left\{\log(x^2+1) + \frac{2x^2}{x^2+1}\right\}dx.$

4. $y = \log\sec^2 x$. *Ans.* $dy = 2\tan x\,dx.$

5. $y = \sin^2 x\cos^2 x$. *Ans.* $dy = \frac{1}{2}\sin 4x\,dx.$

6. $y = \tan^{-1}\sinh x$. *Ans.* $dy = \operatorname{sech} x\,dx.$

7. $y = \frac{1}{3}\tan^3 x + \tan x$. *Ans.* $dy = \sec^4 x\,dx.$

8. $y = \tan^{-1}\log(x+1)$. *Ans.* $\frac{dx}{(x+1)[1+\{\log(x+1)\}^2]}.$

CHAPTER XII

SLOPE OF THE TANGENT LINE IN POLAR COÖRDINATES. SUBTANGENT. SUBNORMAL. ASYMPTOTES

100. We saw in Chapter III that in the curve whose equation is $y = f(x)$ in rectangular coördinates, where $f(x)$ is single valued and continuous, the slope of the tangent line to the curve at any point (x_0, y_0) on the curve is

$$\tan \tau = \left.\frac{dy}{dx}\right|_{x=x_0},$$

or, as we shall write it,

$$\tan \tau_0 = \left.\frac{dy}{dx}\right|_{x=x_0}.$$

To find the slope in polar coördinates of the tangent line to the curve at a point on the curve, we may transform $\left.\frac{dy}{dx}\right|_{x=x_0}$ to polar coördinates by means of the formulas

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Since $x = r \cos \theta$, $\quad \therefore \; dx = \cos \theta \, dr - r \sin \theta \, d\theta.$

Since $y = r \sin \theta$, $\quad \therefore \; dy = \sin \theta \, dr + r \cos \theta \, d\theta.$

$$\therefore \; \frac{dy}{dx} = \frac{\sin \theta \, dr + r \cos \theta \, d\theta}{\cos \theta \, dr - r \sin \theta \, d\theta}$$

$$= \frac{\sin \theta \dfrac{dr}{d\theta} + r \cos \theta}{\cos \theta \dfrac{dr}{d\theta} - r \sin \theta}.$$

Let (r_0, θ_0) be the polar coördinates of the point (x_0, y_0). Therefore

$$\tan \tau_0 = \left.\frac{dy}{dx}\right|_{x=x_0} = \frac{\sin \theta_0 \left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0} + r_0 \cos \theta_0}{\cos \theta_0 \left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0} - r_0 \sin \theta_0}$$

is the slope in polar coördinates of the tangent line to the curve at the point (r_0, θ_0).

If the equation $y = f(x)$ be transformed to polar coördinates, it can be written in the form $r = \phi(\theta)$, or as we shall write it, $r = f(\theta)$, although $f(\theta)$ is not the same function of θ that $f(x)$ is of x.

The slope of the tangent line to the curve $r = f(\theta)$, where $f(\theta)$ is single valued and continuous, at the point (r_0, θ_0), is therefore

$$\tan \tau_0 = \frac{\sin \theta_0 \left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0} + r_0 \cos \theta_0}{\cos \theta_0 \left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0} - r_0 \sin \theta_0}.$$

101. As an illustration of the method of finding the slope of the tangent line to the curve whose equation is expressed in polar coördinates, at a particular point on the curve, consider the following example.

Example. Find the slope of the tangent line to the curve $r = a \sin 2\,\theta$ at the point where $\theta = \frac{\pi}{4}$.

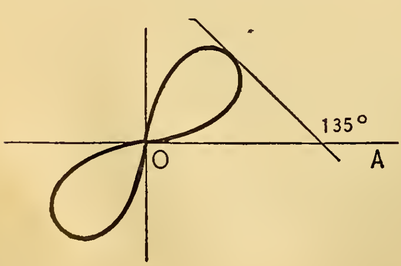


Fig. 31.

By the formula of Art. 100,

$$\tan \tau_0 = \frac{0 \cdot \sin \frac{\pi}{4} + a \cos \frac{\pi}{4}}{0 \cdot \cos \frac{\pi}{4} - a \sin \frac{\pi}{4}} = -1.$$

$$\therefore \tau_0 = 135^\circ.$$

102. The expression for $\tan\tau_0$ in polar coördinates is somewhat complicated. A simpler expression can be found for $\tan\psi_0$, where ψ_0 is the angle between the radius vector to a point and the tangent line to the curve at the point. Let us measure ψ_0 from the positive direction of the radius vector to the tangent line, in anti-clockwise rotation. Then ψ_0 is such that $\tan\psi_0 = \tan(\tau_0 - \theta_0)$ (see Fig. 32).

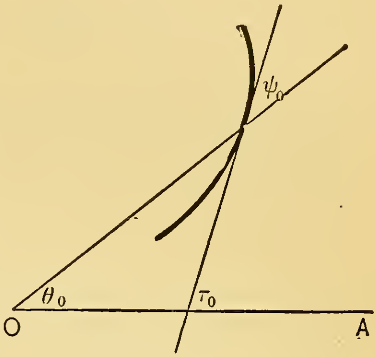


FIG. 32.

$$\therefore \tan\psi_0 = \frac{\tan\tau_0 - \tan\theta_0}{1+\tan\tau_0\tan\theta_0}$$

$$= \frac{\dfrac{\sin\theta_0 \left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0} + r_0\cos\theta_0}{\cos\theta_0 \left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0} - r_0\sin\theta_0} - \dfrac{\sin\theta_0}{\cos\theta_0}}{1 + \dfrac{\sin\theta_0 \left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0} + r_0\cos\theta_0}{\cos\theta_0 \left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0} - r_0\sin\theta_0}\,\dfrac{\sin\theta_0}{\cos\theta_0}}$$

$$= \frac{r_0}{\left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0}}.$$

103. **Subtangent, Subnormal, Rectangular Coördinates.** Let $y = f(x)$ be an equation in which $f(x)$ is single valued and continuous. Let P_0 (Fig. 33) with coördinates (x_0, y_0) be any point on the curve $y = f(x)$. Denote the foot of the ordinate of P_0 by M, and the points where the tangent and normal lines to the curve at P_0 meet the x-axis by T and N respectively.

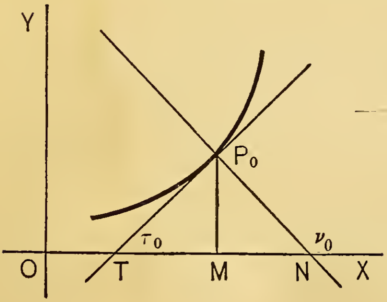


FIG. 33.

Definitions. The lines TM and MN are called the **subtangent** and **subnormal** respectively of the curve at the point P_0.

Since $$\tan\tau_0=\frac{MP_0}{TM},\quad \therefore\ TM=\frac{MP_0}{\tan\tau_0}=\frac{y_0}{\left.\frac{dy}{dx}\right|_{x=x_0}}.$$

The subtangent of the curve at a point (x_0, y_0) on the curve is therefore
$$\frac{y_0}{\left.\frac{dy}{dx}\right|_{x=x_0}}.$$

Denote the angle which the normal line at P_0 makes with the x-axis by ν_0.

Since $$\tan\nu_0=\frac{MP_0}{NM},\quad \therefore\ MN=-\frac{MP_0}{\tan\nu_0}.$$

Now $$\tan\nu_0=\tan\left(\frac{\pi}{2}+\tau_0\right).\quad \therefore\ \tan\nu_0=-\cot\tau_0=-\frac{1}{\left.\frac{dy}{dx}\right|_{x=x_0}}.$$

$$\therefore\ MN=y_0\left.\frac{dy}{dx}\right|_{x=x_0}.$$

The subnormal of the curve at a point (x_0, y_0) on the curve is therefore $$y_0\left.\frac{dy}{dx}\right|_{x=x_0}.$$

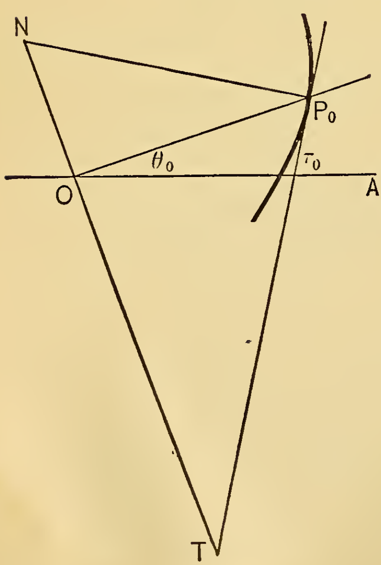


FIG. 34.

104. Subtangent, Subnormal, Polar Coördinates. Let $r=f(\theta)$ be an equation in which $f(\theta)$ is single valued and continuous. Let P_0 (Fig. 34), with coördinates (r_0, θ_0) be a point on the curve $r=f(\theta)$. From the pole O draw a line perpendicular to OP_0 to meet the tangent and normal lines to the curve at P_0 in T and N respectively.

Definitions. The lines OT and ON are called the **polar subtan-**

gent and **polar subnormal** respectively of the curve at the point P_0.

Since $\dfrac{OT}{OP_0} = \tan\psi_0$, $\therefore\; OT = OP_0 \tan\psi_0 = \dfrac{r_0^2}{\left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0}}$.

The polar subtangent of the curve at a point (r_0, θ_0) on the curve is therefore

$$\frac{r_0^2}{\left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0}}.$$

Since $\dfrac{ON}{OP_0} = \tan NP_0O$, $\therefore\; ON = OP_0 \tan NP_0O$.

Now $OP_0 = r_0$, and,

$$\tan NP_0O = \tan\left(\frac{\pi}{2} - \psi_0\right) = \cot\psi_0 = \frac{\left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0}}{r_0}. \quad \therefore\; ON = \left.\frac{dr}{d\theta}\right|_{\theta=\theta_0}.$$

The polar subnormal of the curve at a point (r_0, θ_0) on the curve is therefore

$$\left.\frac{dr}{d\theta}\right|_{\theta=\theta_0}.$$

105. Let $y = f(x)$ be an equation in which $f(x)$ is single valued and has infinite but not finite discontinuities.

If the point of contact of the curve $y = f(x)$ and its tangent line at any point of an infinite branch of the curve is allowed to move so that its distance from the origin increases without limit, the tangent line may or may not approach a limiting position. Thus, as will be seen in Art. 107, the tangent line drawn to a hyperbola thus approaches a limiting position while that drawn to a parabola approaches no limiting position.

106. Definition. If the tangent line to a curve approaches a limiting position as its point of contact with the curve moves along an infinite branch so that its distance from the origin increases without limit, this limiting position is called an **asymptote** of the curve.

The requirements that a curve has an asymptote are two in number, namely:

1st. That the curve has an infinite branch.

2d. That the tangent line at a point on an infinite branch of the curve approaches a limiting position as its point of contact moves along the branch so that its distance from the origin increases without limit.

If the intercepts of the tangent line, or one intercept and the slope, approach limits as the point of contact of the tangent line and the curve moves on an infinite branch so that its distance from the origin increases without limit, these limits determine the equation of the asymptote. If they do not approach limits, this branch of the curve has no asymptote.

107. The equation of the tangent line to the curve $y=f(x)$ at a point (x_0, y_0) on the curve is

$$y-y_0=\left.\frac{dy}{dx}\right|_{x=x_0}(x-x_0).$$

Therefore its intercepts are:

$$x_0-y_0\frac{1}{\left.\dfrac{dy}{dx}\right|_{x=x_0}}, \text{ intercept on the } x\text{-axis};$$

$$y_0-x_0\left.\frac{dy}{dx}\right|_{x=x_0}, \text{ intercept on the } y\text{-axis}.$$

Example 1. Investigate the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ for asymptotes.

$$\left.\frac{dy}{dx}\right|_{x=x_0}=\frac{b^2x_0}{a^2y_0}.$$

$$\therefore\ x_0-y_0\frac{1}{\left.\dfrac{dy}{dx}\right|_{x=x_0}}=x_0-\frac{a^2y_0^2}{b^2x_0}=\frac{b^2x_0^2-a^2y_0^2}{b^2x_0},$$

and

$$y_0-x_0\left.\frac{dy}{dx}\right|_{x=x_0}=y_0-\frac{b^2x_0^2}{a^2y_0}=\frac{a^2y_0^2-b^2x_0^2}{a^2y_0}.$$

Simplify by means of the equality $\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}=1$.

$$\therefore \quad \frac{b^2x_0^2-a^2y_0^2}{b^2x_0}=\frac{a^2}{x_0}, \quad \text{and} \quad \frac{a^2y_0^2-b^2x_0^2}{a^2y_0}=-\frac{b^2}{y_0}.$$

As the point of contact of the tangent line moves along the curve so that its distance from the origin increases without limit, the coördinates both increase without limit. If (x_0, y_0) are the coördinates of the moving point,

$$\underset{x_0=\infty}{\text{limit}}\left[\frac{a^2}{x_0}\right]=0, \quad \text{and} \quad \underset{y_0=\infty}{\text{limit}}\left[-\frac{b^2}{y_0}\right]=0.$$

Therefore the hyperbola has an asymptote.

$$\text{Since} \qquad \left.\frac{dy}{dx}\right|_{x=x_0}=\frac{b^2x_0}{a^2y_0}$$

$$=\pm\frac{bx_0}{a\sqrt{x_0^2-a^2}}=\pm\frac{b}{a}\frac{1}{\sqrt{1-\frac{a^2}{x_0^2}}},$$

$$\therefore \quad \underset{x_0=\infty}{\text{limit}}\left[\left.\frac{dy}{dx}\right|_{x=x_0}\right]=\underset{x_0=\infty}{\text{limit}}\left[\pm\frac{b}{a}\frac{1}{\sqrt{1-\frac{a^2}{x_0^2}}}\right]=\pm\frac{b}{a}.$$

The hyperbola has therefore two asymptotes, one with slope $+\frac{b}{a}$ and the other with slope $-\frac{b}{a}$. Their equations can be written readily from analytic geometry. They are:

$$\frac{x}{a}-\frac{y}{b}=0,$$

and

$$\frac{x}{a}+\frac{y}{b}=0.$$

Example 2. Investigate the parabola $y^2=2\,mx$ for asymptotes.

$$\left.\frac{dy}{dx}\right|_{x=x_0}=\frac{m}{y_0}.$$

$$\therefore\ x_0 - y_0 \frac{1}{\left.\dfrac{dy}{dx}\right|_{x=x_0}} = x_0 - \frac{y_0^2}{m} = -x_0,$$

and
$$y_0 - x_0 \left.\frac{dy}{dx}\right|_{x=x_0} = y_0 - \frac{mx_0}{y_0} = \frac{y_0}{2}.$$

Now
$$\underset{x_0=\infty}{\text{limit}}\left[-x_0\right] = -\infty, \text{ and } \underset{y_0=\infty}{\text{limit}}\left[\frac{y_0}{2}\right] = \infty.$$

Therefore neither intercept approaches a limit, and therefore the parabola has no asymptote.

108. If the polar subtangent OT (Fig. 34) approaches a limit as the point P_0 moves along an infinite branch so that its distance from the pole increases without limit, the curve has an asymptote passing through T in the limiting position of OT and parallel to OP_0.

Example. Investigate the curve $r = a \sec 2\theta$ for asymptotes.

Since $r = a \sec 2\theta$,

$$\therefore\ \frac{dr}{d\theta} = 2a \sec 2\theta \tan 2\theta.$$

$$\therefore\ \frac{r^2}{\dfrac{dr}{d\theta}} = \frac{a}{2}\frac{\sec 2\theta}{\tan 2\theta} = \frac{a}{2}\operatorname{cosec} 2\theta.$$

As the point of contact of the tangent line moves along the curve so that its distance from the pole increases without limit, θ approaches either $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, or $\frac{7\pi}{4}$. Now

$$\underset{\theta \doteq \frac{\pi}{4}}{\text{limit}}\left[\frac{a}{2}\operatorname{cosec} 2\theta\right] = \frac{a}{2}; \qquad \underset{\theta \doteq \frac{3\pi}{4}}{\text{limit}}\left[\frac{a}{2}\operatorname{cosec} 2\theta\right] = -\frac{a}{2};$$

$$\underset{\theta \doteq \frac{5\pi}{4}}{\text{limit}}\left[\frac{a}{2}\operatorname{cosec} 2\theta\right] = -\frac{a}{2}; \qquad \underset{\theta \doteq \frac{7\pi}{4}}{\text{limit}}\left[\frac{a}{2}\operatorname{cosec} 2\theta\right] = \frac{a}{2}.$$

The curve has therefore four asymptotes, two intersecting on the initial line at a distance $\frac{a}{\sqrt{2}}$ from the pole and making angles of 45° and 315° respectively with the initial line, and two intersecting on the initial line at a distance $-\frac{a}{\sqrt{2}}$ from the pole and making angles of 135° and 225° respectively with the initial line.

The curve is as in Fig. 35.

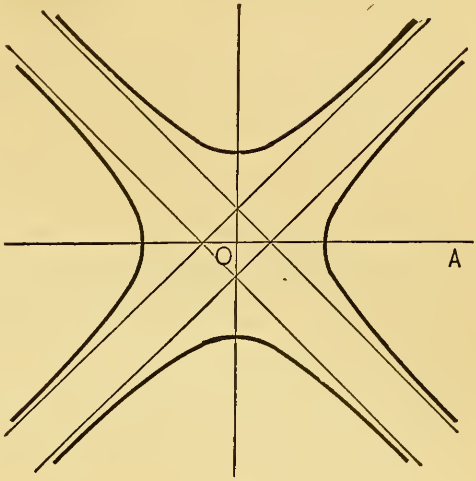


FIG. 35.

EXERCISES

1. Find the slope of the cardioid $r = a(1 - \cos\theta)$ when $\theta = \frac{\pi}{2}$. *Ans.* -1.

2. Find the slope of the curve $r = a\sec^2\theta$ when $r = 2a$. *Ans.* 3.

3. In the circle $r = a\cos\theta$, find τ_0 and ψ_0.

Ans. If θ varies between 0 and π,

$$\tau_0 = \pm\frac{\pi}{2} + 2\theta_0, \text{ or } -\frac{3\pi}{2} + 2\theta_0;\ \psi_0 = \pm\frac{\pi}{2} + \theta_0.$$

4. In the logarithmic spiral $r = a^{n\theta}$, show that ψ is constant.

5. In the lemniscate $r^2 = a^2\sin 2\theta$, find τ_0 and ψ_0.

Ans. If θ varies between 0 and $\frac{\pi}{2}$,

$$\tau_0 = 3\theta_0, \text{ or } -\pi + 3\theta_0;\ \psi_0 = 2\theta_0.$$

In each of the three following curves, find the subtangent and subnormal at any point (x_0, y_0) on the curve.

6. $y^2 = 4ax$. *Ans.* $2x_0$; $2a$.

7. $xy = c$. *Ans.* $-x_0$; $-\frac{y_0^3}{c}$.

8. $y^2 = \dfrac{x^3}{2a - x}$. *Ans.* $\dfrac{2ax_0 - x_0^2}{3a - x_0}$; $\dfrac{3ax_0^2 - x_0^3}{(2a - x_0)^2}$.

9. In the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, find the subtangent and subnormal for a value θ_0 of θ.

Ans. $2a\sin^3\dfrac{\theta_0}{2}\sec\dfrac{\theta_0}{2}$; $a\sin\theta_0$.

10. In the curve $x = a\log\cot\dfrac{\theta}{2} - a\cos\theta$, $y = a\sin\theta$, find the subtangent for a value θ_0 of θ. *Ans.* $-a\cos\theta_0$.

11. In the lemniscate $r^2 = a^2\sin 2\theta$, find the polar subtangent for a value θ_0 of θ. *Ans.* $a\tan 2\theta_0\sqrt{\sin 2\theta_0}$.

12. In the curve $r = a\sec^2\theta$, find the polar subnormal for a value θ_0 of θ. *Ans.* $2a\sec^2\theta_0\tan\theta_0$.

Investigate the following curves for asymptotes.

13. $y^2 = x^2 - 3x - 4$. *Ans.* Asymptotes, $2x - 2y = 3$, $2x + 2y = 3$.

14. $y^3 = x^3 - 8x^2$. *Ans.* Asymptote, $x - y = \frac{8}{3}$.

15. $y^2 = \dfrac{x^3}{2a - x}$. *Ans.* Asymptote, $x = 2a$.

16. $r = a\tan\theta$. *Ans.* Two asymptotes perpendicular to OA and distant a from O.

17. $r = \dfrac{a}{\theta}$. *Ans.* An asymptote parallel to OA at a distance a above.

CHAPTER XIII

CURVATURE. RADIUS OF CURVATURE. EVOLUTES AND INVOLUTES

109. Definitions. The **integral curvature** of an arc of a curve whose equation gives a continuous function is the angle between the tangents at the extremities of the arc.

Thus, if P_0P (Fig. 36) be the arc, and AP_0 and BP the tangents at its extremities, the integral curvature is the angle ACB.

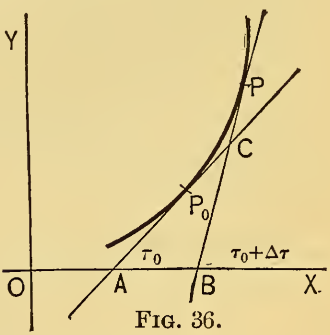


FIG. 36.

The **mean curvature** of an arc of a curve whose equation gives a continuous function is the integral curvature of the arc divided by the arc, the integral curvature being measured in units of angle and the arc in units of length.

Thus, in Fig. 36, the mean curvature is $\dfrac{\angle ACB}{\text{arc } P_0P}$.

Mean curvature is usually expressed in radians per unit of length.

The **actual curvature** at a point on a curve whose equation gives a continuous function is the limit which the mean curvature of an arc beginning at the point approaches as the arc is allowed to decrease without limit.

Thus, in Fig. 36, the actual curvature of the curve at the point P_0 is $\underset{\text{arc } P_0P = 0}{\text{limit}} \left[\dfrac{\angle ACB}{\text{arc } P_0P}\right]$.

For brevity, actual curvature is spoken of merely as curvature.

The curvature of an arc is said to be **uniform** when it is the same at all points of the arc.

110. Let $y = f(x)$ be an equation in which $f(x)$ is single valued and continuous. To find the curvature at a point (x_0, y_0) on the curve $y = f(x)$.

Suppose that the curve is as in Fig. 37. Let P_0 be the point whose coördinates are (x_0, y_0). Let $x = x_0 + \Delta x$. Let P be the point whose abscissa is $x_0 + \Delta x$. Denote the arc P_0P by Δs and the chord P_0P by Δc.

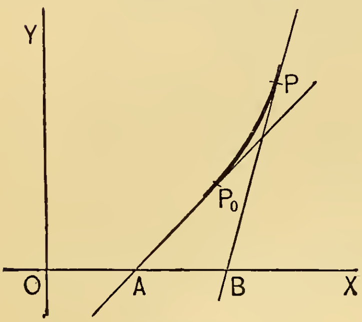


Fig. 37.

Denote the angle which the tangent line at P_0 makes with the x-axis by τ_0. The angle which the tangent line at P makes with the x-axis is therefore $\tau_0 + \Delta\tau$, where $\Delta\tau$ is the increment in the angle due to x having taken the increment Δx.

$$\text{The curvature at } P_0 = \underset{\Delta s \doteq 0}{\text{limit}}\left[\frac{\Delta\tau}{\Delta s}\right]$$

$$= \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta\tau}{\Delta s}\right], \text{ since } \Delta x \doteq 0 \text{ as } \Delta s \doteq 0.$$

It will be shown in Chapter XXII that the limit of the ratio of a chord to its arc as both approach zero is 1. Assuming this theorem, we have:

$$\text{The curvature at } P_0 = \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta\tau}{\Delta c}\cdot\frac{\Delta c}{\Delta s}\right]$$

$$= \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta\tau}{\Delta c}\right]\cdot \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta c}{\Delta s}\right]$$

$$= \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta\tau}{\Delta c}\right], \text{ since } \underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta c}{\Delta s}\right] = 1,$$

$$= \lim_{\Delta x \doteq 0} \left[\frac{\Delta \tau}{\sqrt{\Delta x^2 + \Delta y^2}} \right]$$

$$= \lim_{\Delta x \doteq 0} \left[\frac{\frac{\Delta \tau}{\Delta x}}{\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}} \right],$$

by dividing and multiplying by Δx,

$$= \frac{\left.\frac{d\tau}{dx}\right|_{x=x_0}}{\sqrt{1 + \left(\left.\frac{dy}{dx}\right|_{x=x_0}\right)^2}}. \tag{1}$$

To express $\left.\frac{d\tau}{dx}\right|_{x=x_0}$ in terms of x and y.

$$\tan \tau = \frac{dy}{dx}.$$

$$\therefore \sec^2 \tau \frac{d\tau}{dx} = \frac{d^2y}{dx^2}.$$

$$\therefore \left.\frac{d\tau}{dx}\right|_{x=x_0} = \frac{\left.\frac{d^2y}{dx^2}\right|_{x=x_0}}{\sec^2 \tau_0} = \frac{\left.\frac{d^2y}{dx^2}\right|_{x=x_0}}{1 + \tan^2 \tau_0}$$

$$= \frac{\left.\frac{d^2y}{dx^2}\right|_{x=x_0}}{1 + \left(\left.\frac{dy}{dx}\right|_{x=x_0}\right)^2}.$$

Substitute this value in (1).

Therefore the curvature at $P_0 = \dfrac{\left.\frac{d^2y}{dx^2}\right|_{x=x_0}}{\left\{1 + \left(\left.\frac{dy}{dx}\right|_{x=x_0}\right)^2\right\}^{\frac{3}{2}}}$. (2)

Curvature is usually denoted by κ.

111. In the expression (2) for curvature, in the preceding article, the positive square root was taken. Then, since the denominator is always positive, the sign of κ will depend on that of $\left.\frac{d^2y}{dx^2}\right|_{x=x_0}$. Therefore κ is positive if the curve is concave upwards, and negative if concave downwards.

There is no advantage in distinguishing between positive and negative curvature, and consequently, when the curvature is negative, we shall omit the minus sign and consider it as positive.

112. As an illustration of the method of finding the curvature at a given point on a given curve, consider the following example.

Example. Find the curvature at a point (x_0, y_0) on the circle $x^2 + y^2 = a^2$.

Since $$x^2 + y^2 = a^2, \therefore\ y = \pm\sqrt{a^2 - x^2}.$$

At first consider the part of the curve above the x-axis.

Then $$y = +\sqrt{a^2 - x^2}.$$

$$\therefore\ \frac{dy}{dx} = -\frac{x}{\sqrt{a^2 - x^2}}, \text{ and } \frac{d^2y}{dx^2} = -\frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

$$\therefore\ \kappa = \frac{-\dfrac{a^2}{(a^2 - {x_0}^2)^{\frac{3}{2}}}}{\left\{1 + \dfrac{{x_0}^2}{a^2 - {x_0}^2}\right\}^{\frac{3}{2}}} = -\frac{1}{a}.$$

Or, as we shall say, $\kappa = \frac{1}{a}$.

Next consider the part of the curve below the x-axis.

Then $$y = -\sqrt{a^2 - x^2}.$$

$$\therefore \frac{dy}{dx} = \frac{x}{\sqrt{a^2 - x^2}}, \text{ and } \frac{d^2y}{dx^2} = \frac{a^2}{(a^2 - x^2)^{\frac{3}{2}}}.$$

$$\therefore \kappa = \frac{\dfrac{a^2}{(a^2 - x_0^{\,2})^{\frac{3}{2}}}}{\left\{1 + \dfrac{x_0^{\,2}}{a^2 - x_0^{\,2}}\right\}^{\frac{3}{2}}} = \frac{1}{a}.$$

113. Let $r = f(\theta)$ be an equation in which $f(\theta)$ is single valued and continuous. To find the curvature at a point (r_0, θ_0) on the curve $r = f(\theta)$.

Suppose that the curve is as in Fig. 38. Let P_0 be the point whose coördinates are (r_0, θ_0). Let $\theta = \theta_0 + \Delta\theta$. Let P be the point at which $\theta = \theta_0 + \Delta\theta$. Denote the arc P_0P by Δs and the chord P_0P by Δc. Let the angles which the tangent lines at P_0 and P make with the initial line be denoted by τ_0 and $\tau_0 + \Delta\tau$ respectively.

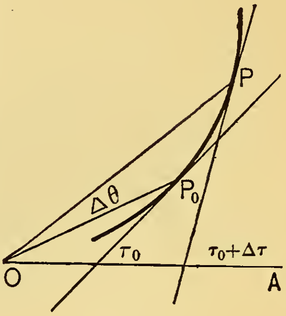


FIG. 38.

Suppose that the equation $r = f(\theta)$ is transformed to rectangular coördinates. The equations of transformation are

$$x = r \cos \theta,$$

$$y = r \sin \theta.$$

If P_0 is the point whose abscissa is x_0, and P the point whose abscissa is $x_0 + \Delta x$,

$$\kappa = \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta\tau}{\Delta s}\right]$$

$$= \underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta\tau}{\sqrt{\Delta x^2 + \Delta y^2}}\right].$$

Now $\Delta\theta \doteq 0$ as $\Delta x \doteq 0$.

Therefore $$\kappa = \underset{\Delta\theta \doteq 0}{\text{limit}} \left[\frac{\Delta\tau}{\sqrt{\Delta x^2 + \Delta y^2}} \right]$$

$$= \underset{\Delta\theta \doteq 0}{\text{limit}} \left[\frac{\dfrac{\Delta\tau}{\Delta\theta}}{\sqrt{\left(\dfrac{\Delta x}{\Delta\theta}\right)^2 + \left(\dfrac{\Delta y}{\Delta\theta}\right)^2}} \right],$$

by dividing and multiplying by $\Delta\theta$,

$$= \frac{\left.\dfrac{d\tau}{d\theta}\right|_{\theta=\theta_0}}{\sqrt{\left(\left.\dfrac{dx}{d\theta}\right|_{\theta=\theta_0}\right)^2 + \left(\left.\dfrac{dy}{d\theta}\right|_{\theta=\theta_0}\right)^2}}. \tag{1}$$

Since $\tan\tau = \tan(\theta + \psi)$ (see Art. 102),

$\therefore\ \tau = k\pi + \theta + \psi$, where k is zero or an integer positive or negative.

$$\therefore\ \frac{d\tau}{d\theta} = 1 + \frac{d\psi}{d\theta}.$$

Since $\tan\psi = \dfrac{r}{\dfrac{dr}{d\theta}}$ (see Art. 102),

$$\therefore\ \psi = \tan^{-1} \frac{r}{\dfrac{dr}{d\theta}}.$$

$$\therefore\ \frac{d\psi}{d\theta} = \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}.$$

$$\therefore\ \left.\frac{d\tau}{d\theta}\right|_{\theta=\theta_0} = \frac{r_0^2 + 2\left(\left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0}\right)^2 - r_0\left.\dfrac{d^2r}{d\theta^2}\right|_{\theta=\theta_0}}{r_0^2 + \left(\left.\dfrac{dr}{d\theta}\right|_{\theta=\theta_0}\right)^2}.$$

Also, since $x = r\cos\theta$, $\therefore\ dx = \cos\theta\, dr - r\sin\theta\, d\theta$; and since $y = r\sin\theta$, $\therefore\ dy = \sin\theta\, dr + r\cos\theta\, d\theta$.

$$\therefore \sqrt{\left(\frac{dx}{d\theta}\bigg|_{\theta=\theta_0}\right)^2 + \left(\frac{dy}{d\theta}\bigg|_{\theta=\theta_0}\right)^2}$$

$$= \sqrt{\left(\cos\theta_0 \frac{dr}{d\theta}\bigg|_{\theta=\theta_0} - r_0\sin\theta_0\right)^2 + \left(\sin\theta_0\frac{dr}{d\theta}\bigg|_{\theta=\theta_0} + r_0\cos\theta_0\right)^2}$$

$$= \sqrt{r_0^2 + \left(\frac{dr}{d\theta}\bigg|_{\theta=\theta_0}\right)^2}.$$

Substitute the value of

$$\frac{d\tau}{d\theta}\bigg|_{\theta=\theta_0}, \text{ and } \sqrt{\left(\frac{dx}{d\theta}\bigg|_{\theta=\theta_0}\right)^2 + \left(\frac{dy}{d\theta}\bigg|_{\theta=\theta_0}\right)^2} \text{ in (1).}$$

$$\therefore\ \kappa = \frac{r_0^2 + 2\left(\dfrac{dr}{d\theta}\bigg|_{\theta=\theta_0}\right)^2 - r_0\dfrac{d^2r}{d\theta^2}\bigg|_{\theta=\theta_0}}{\left[r_0^2 + \left(\dfrac{dr}{d\theta}\bigg|_{\theta=\theta_0}\right)^2\right]^{\frac{3}{2}}}.$$

114. As was seen in Art. 112, the curvature of a circle is the reciprocal of the radius of the circle. Then, since a circle can be described with any radius, a circle can be described to have any curvature.

115. Definitions. Two curves are said to be tangent to each other at a point when they coincide at the point and have the same tangent line there.

The circle tangent to the curve and having the same curvature as the curve at a point, whose concavity has the same aspect as that of the curve, is called the **osculating circle** of the curve at the point.

The radius of the osculating circle at a point on a curve is called the **radius of curvature** of the curve at the point.

The center of the osculating circle at a point on a curve is called the **center of curvature** of the curve at the point.

From its definition, the radius of curvature of a curve at a point is the reciprocal of the curvature at the point.

EVOLUTES

116. Definition. The locus of the center of curvature of a given curve is called the **evolute** of the curve.

Let $y=f(x)$ be an equation in which $f(x)$ is single valued and continuous. To find the evolute of the curve $y=f(x)$.

Suppose that the curve is concave upwards and rising, as in Fig. 39. From the investigation in this case, the student can readily investigate the other cases for himself. It will be found that in all cases the results are the same.

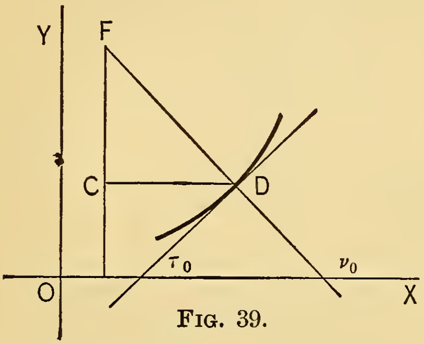


FIG. 39.

Let D with coördinates (x_0, y_0) be a point on the curve, and F with coördinates (x', y') the corresponding center of curvature. Denote the radius of curvature of the curve at D by ρ. From D draw DC parallel to the x-axis to meet the ordinate y' in C.

Then $$x' = x_0 - CD,$$

and $$y' = y_0 + CF.$$

Since $$\nu_0 = \tau_0 + \frac{\pi}{2}, \quad \therefore \tan \nu_0 = -\frac{1}{\left.\frac{dy}{dx}\right|_{x=x_0}}.$$

$$\therefore \cos \nu_0 = \frac{-\left.\frac{dy}{dx}\right|_{x=x_0}}{\sqrt{1+\left(\left.\frac{dy}{dx}\right|_{x=x_0}\right)^2}}, \text{ and } \sin \nu_0 = \frac{1}{\sqrt{1+\left(\left.\frac{dy}{dx}\right|_{x=x_0}\right)^2}}.$$

Now $CD = |\rho \cos \nu_0| = \left| \dfrac{\left\{1+\left(\left.\dfrac{dy}{dx}\right|_{x=x_0}\right)^2\right\}^{\frac{3}{2}}}{\left.\dfrac{d^2y}{dx^2}\right|_{x=x_0}} \cdot \dfrac{-\left.\dfrac{dy}{dx}\right|_{x=x_0}}{\sqrt{1+\left(\left.\dfrac{dy}{dx}\right|_{x=x_0}\right)^2}} \right|$

$$= \left.\frac{dy}{dx}\right|_{x=x_0} \cdot \frac{1+\left(\left.\dfrac{dy}{dx}\right|_{x=x_0}\right)^2}{\left.\dfrac{d^2y}{dx^2}\right|_{x=x_0}},$$

because $\left.\dfrac{dy}{dx}\right|_{x=x_0}$ is positive since the curve is rising, and $\left.\dfrac{d^2y}{dx^2}\right|_{x=x_0}$ is positive since the curve is concave upwards.

$$\therefore\ x' = x_0 - \left.\frac{dy}{dx}\right|_{x=x_0} \cdot \frac{1+\left(\left.\dfrac{dy}{dx}\right|_{x=x_0}\right)^2}{\left.\dfrac{d^2y}{dx^2}\right|_{x=x_0}}. \qquad (1)$$

Also, $\qquad y' = y_0 + |\rho \sin \nu_0|.$

$$\therefore\ y' = y_0 + \frac{1+\left(\left.\dfrac{dy}{dx}\right|_{x=x_0}\right)^2}{\left.\dfrac{d^2y}{dx^2}\right|_{x=x_0}}. \qquad (2)$$

If x_0 and y_0 be eliminated from equations (1), (2), and $y_0 = f(x_0)$, the resulting equation is that of the evolute.

117. Example 1. Find the evolute of the parabola $y^2 = 2\,mx$.

At first take $y = +\sqrt{2\,m}\sqrt{x}$ and consider the part of the curve above the x-axis.

$$\frac{dy}{dx} = \sqrt{\frac{m}{2}}\,\frac{1}{\sqrt{x}}.$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\sqrt{\frac{m}{2}}\,\frac{1}{x^{\frac{3}{2}}}.$$

$$\therefore\ x' = x_0 - \sqrt{\frac{m}{2}}\,\frac{1}{\sqrt{x_0}}\,\frac{1+\dfrac{m}{2\,x_0}}{-\dfrac{1}{2}\sqrt{\dfrac{m}{2}}\,\dfrac{1}{x_0^{\frac{3}{2}}}}$$

$$= 3\,x_0 + m,$$

and
$$y' = y_0 + \frac{1+\dfrac{m}{2\,x_0}}{-\dfrac{1}{2}\sqrt{\dfrac{m}{2}}\,\dfrac{1}{x_0^{\frac{3}{2}}}}$$

$$= -\frac{y_0^3}{m^2}. \tag{1}$$

$$\therefore\ x_0 = \frac{x'-m}{3},\ \text{and}\ y_0 = -\sqrt[3]{m^2y'}.$$

Substitute the values of x_0 and y_0 in $y_0^2 = 2\,mx_0$.

$\therefore\ my'^2 = \frac{8}{27}(x'-m)^3$ is the equation of the evolute. (2)

Solve equation (2) for y'. $\therefore\ y' = \pm\dfrac{2\sqrt{2}}{3\sqrt{3\,m}}(x'-m)^{\frac{3}{2}}$.

In this equation the minus sign must be taken, since, from equation (1), y' is negative. The part of the parabola above the x-axis and its evolute are therefore as in Fig. 40.

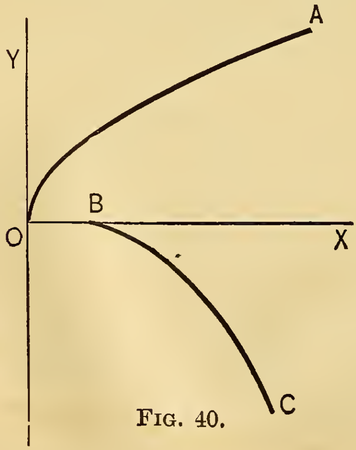


FIG. 40.

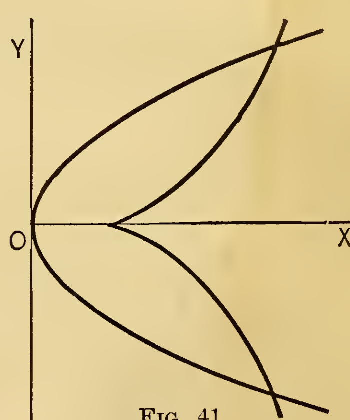


FIG. 41.

Similarly we may show that the evolute of the part of the parabola below the x-axis has the equation $y' = +\frac{2\sqrt{2}}{3\sqrt{3m}}(x'-m)^{\frac{3}{2}}$. The parabola and entire evolute are as in Fig. 41.

EXAMPLE 2. Find the evolute of the cycloid

$$x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta).$$

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right)\frac{d\theta}{dx}$$

$$= -\frac{1}{a(1-\cos\theta)^2}.$$

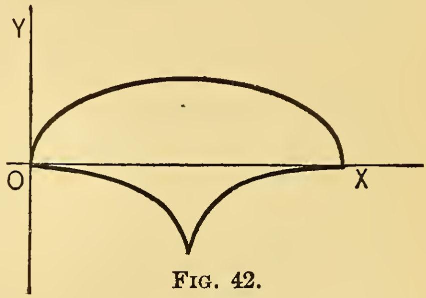


FIG. 42.

$$\therefore \quad x' = a(\theta_0 - \sin\theta_0) + 2a\sin\theta_0 = a(\theta_0 + \sin\theta_0).$$

$$y' = a(1-\cos\theta_0) - 2a(1-\cos\theta_0) = -a(1-\cos\theta_0).$$

Therefore the equations of the evolute are

$$x' = a(\theta + \sin\theta),$$

$$y' = -a(1-\cos\theta).$$

One arch of the cycloid, and its evolute, are as in Fig. 42.

PROPERTY OF THE EVOLUTE

118. The evolute of a curve has the property that the extremity of a stretched string unwound from it traces out the original curve.

Thus, in Fig. 40, Art. 117, the extremity O of a stretched string OBC unwound from BC traces out the part of the parabola OA.

On account of the property of the evolute just mentioned the original curve is called the **involute**.

119. To prove the above property of the evolute we have to establish two theorems.

Theorem I. Every normal to the involute is tangent to the evolute.

Theorem II. The length of any arc of the evolute is equal to the difference between the lengths of the radii of curvature of the involute which pass through the extremities of the arc in question.

Proof of Theorem I.

Let D with coördinates (x, y) be a point on the involute, and F with coördinates (x', y'), the corresponding point on the evolute. (Fig. 43.)

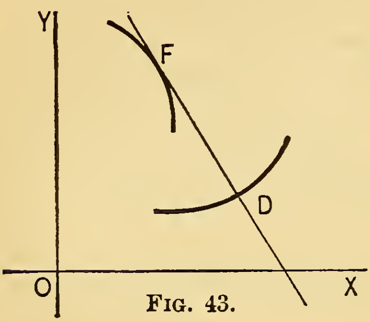


Fig. 43.

Let the tangent line at F and the normal line at D make the angles τ' and ν respectively with the x-axis. The theorem is proved when it is shown that $\tan\tau' = \tan\nu$.

$$\tan\tau' = \frac{dy'}{dx'} = \frac{\frac{dy'}{dx}}{\frac{dx'}{dx}}.$$

From (1) and (2) of Art. 116,

$$x' = x - \frac{dy}{dx} \cdot \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.$$

$$y' = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.$$

$$\therefore \quad \frac{dy'}{dx} = \frac{3\dfrac{dy}{dx}\left(\dfrac{d^2y}{dx^2}\right)^2 - \dfrac{d^3y}{dx^3} - \left(\dfrac{dy}{dx}\right)^2\dfrac{d^3y}{dx^3}}{\left(\dfrac{d^2y}{dx^2}\right)^2},$$

and
$$\frac{dx'}{dx} = -\frac{\dfrac{dy}{dx}\left\{3\dfrac{dy}{dx}\left(\dfrac{d^2y}{dx^2}\right)^2 - \dfrac{d^3y}{dx^3} - \left(\dfrac{dy}{dx}\right)^2\dfrac{d^3y}{dx^3}\right\}}{\left(\dfrac{d^2y}{dx^2}\right)^2}.$$

$$\therefore \quad \frac{\dfrac{dy'}{dx}}{\dfrac{dx'}{dx}} = -\frac{1}{\dfrac{dy}{dx}}, \text{ by division.}$$

$$\therefore \quad \tan\tau' = \tan\nu.$$

Proof of Theorem II.

Let D with coördinates (x, y) be a point on the involute, and F with coördinates (x', y') the corresponding point on the evolute. Let x take an increment Δx. Let E be the point on the involute whose coördinates are $(x+\Delta x, y+\Delta y)$, and G the corresponding point on the evolute. Let $(x'+\Delta x', y'+\Delta y')$ denote the coördinates of G. Denote the arc FG by $\Delta s'$, and the chord FG by $\Delta c'$.

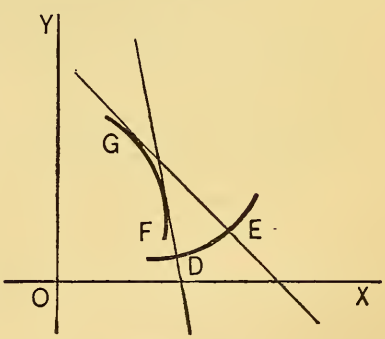


FIG. 44.

Now $\quad \dfrac{\Delta s'}{\Delta x} = \dfrac{\Delta c'}{\Delta x} \cdot \dfrac{\Delta s'}{\Delta c'}$, identically.

$$\therefore \lim_{\Delta x \doteq 0}\left[\frac{\Delta s'}{\Delta x}\right] = \lim_{\Delta x \doteq 0}\left[\frac{\Delta c'}{\Delta x}\right] \cdot \lim_{\Delta x \doteq 0}\left[\frac{\Delta s'}{\Delta c'}\right]$$

$$= \lim_{\Delta x \doteq 0}\left[\frac{\Delta c'}{\Delta x}\right], \text{ since } \lim_{\Delta x \doteq 0}\left[\frac{\Delta s'}{\Delta c'}\right] = 1 \text{ (see Art. 194)}$$

$$= \lim_{\Delta x \doteq 0}\left[\frac{\sqrt{\Delta x'^2 + \Delta y'^2}}{\Delta x}\right].$$

$$\therefore \frac{ds'}{dx} = \sqrt{\left(\frac{dx'}{dx}\right)^2 + \left(\frac{dy'}{dx}\right)^2}$$

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, \frac{3\dfrac{dy}{dx}\left(\dfrac{d^2y}{dx^2}\right)^2 - \dfrac{d^3y}{dx^3} - \left(\dfrac{dy}{dx}\right)^2 \dfrac{d^3y}{dx^3}}{\left(\dfrac{d^2y}{dx^2}\right)^2}.$$

From the formula of Art. 110,

$$\rho = \frac{1}{\kappa} = \frac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}.$$

$$\therefore \frac{d\rho}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, \frac{3\dfrac{dy}{dx}\left(\dfrac{d^2y}{dx^2}\right)^2 - \dfrac{d^3y}{dx^3} - \left(\dfrac{dy}{dx}\right)^2 \dfrac{d^3y}{dx^3}}{\left(\dfrac{d^2y}{dx^2}\right)^2}.$$

$$\therefore \frac{ds'}{dx} = \frac{d\rho}{dx}.$$

It will be seen later that since $\dfrac{ds'}{dx} = \dfrac{d\rho}{dx}$, $\therefore s' = \rho + c$, where c is any constant. The student can see now that if $s' = \rho + c$ be differentiated with respect to x, the result is $\dfrac{ds'}{dx} = \dfrac{d\rho}{dx}$.

Let s_1' and s_2' be the lengths of arcs of the evolute measured from some fixed point on it, and ρ_1 and ρ_2 the radii of curvature, which pass through the extremities of these arcs.

Then, since $\qquad s' = \rho + c,$

we have $\qquad s_1' = \rho_1 + c,$

and $\qquad s_2' = \rho_2 + c.$

$$\therefore s_1' - s_2' = \rho_1 - \rho_2.$$

Now $s_1' - s_2'$ is any length of arc, and $\rho_1 - \rho_2$ is the difference between the radii of curvature which pass through the extremities of the arc. The theorem is therefore proved.

EXERCISES

In each of the two following curves, find the curvature at a point (x_0, y_0) on the curve.

1. $y^2 = 2\,mx$. *Ans.* $\kappa = \dfrac{m^2}{\{m^2 + y_0^2\}^{\frac{3}{2}}}$.

2. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. *Ans.* $\kappa = \dfrac{a^4 b^4}{\{b^4 x_0^2 + a^4 y_0^2\}^{\frac{3}{2}}}$.

In each of the two following curves, find the curvature at a point (r_0, θ_0) on the curve.

3. $r = a(1 - \cos\theta)$. *Ans.* $\kappa = \dfrac{3}{2\sqrt{2\,ar_0}}$.

4. $r = a \sec^2 \dfrac{\theta}{2}$. *Ans.* $\kappa = \dfrac{1}{2\,a} \cos^3 \dfrac{\theta_0}{2}$.

In each of the six following curves, find the radius of curvature, ρ, at a point (x_0, y_0) on the curve.

5. $xy = c$. *Ans.* $\rho = \dfrac{(x_0^2 + y_0^2)^{\frac{3}{2}}}{2\,c}$.

6. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. *Ans.* $\rho = \dfrac{\{b^4 x_0^2 + a^4 y_0^2\}^{\frac{3}{2}}}{a^4 b^4}$.

7. $y = \dfrac{a}{2}(e^{\frac{x}{a}} + e^{-\frac{x}{a}})$. *Ans.* $\rho = \dfrac{y_0^2}{a}$.

8. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\rho = 3(ax_0 y_0)^{\frac{1}{3}}$.

9. $e^x = \sin y$. *Ans.* $\rho = e^{-x_0}$.

10. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$. *Ans.* $\rho = \dfrac{2\,(x_0 + y_0)^{\frac{3}{2}}}{\sqrt{a}}$.

11. Find the radius of curvature, ρ, of the cycloid

$x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ for a value θ_0 of θ.

Ans. $\rho = 4\,a \sin \dfrac{\theta_0}{2}$.

12. Find the radii of curvature of the trochoid $x = a\theta + k\sin\theta$, $y = a - k\cos\theta$ at the points where it is nearest to and farthest from the base. *Ans.* $\dfrac{(a \pm k)^2}{k}$.

In each of the three following equations find the radius of curvature, ρ, at a point (r_0, θ_0) on the curve:

13. $r^2 = a^2 \cos 2\theta$. *Ans.* $\rho = \dfrac{a^2}{3r_0}$.

14. $r^2 \cos 2\theta = a^2$. *Ans.* $\rho = \dfrac{r_0^3}{a^2}$.

15. $r(1+\cos\theta) = 2a$. *Ans.* $\rho = \dfrac{2r_0^{\frac{3}{2}}}{\sqrt{a}}$.

In each of the six following curves, find the equation of the evolute of the curve:

16. $x^2 + y^2 = a^2$. *Ans.* $x' = 0,\ y' = 0$.

17. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. *Ans.* $(ax')^{\frac{2}{3}} + (by')^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

18. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. *Ans.* $(ax')^{\frac{2}{3}} - (by')^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}$.

19. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$. *Ans.* $\left(\dfrac{x'+y'}{a}\right)^{\frac{2}{3}} + \left(\dfrac{x'-y'}{a}\right)^{\frac{2}{3}} = 2$.

20. $xy = c$. *Ans.* $(x'+y')^{\frac{2}{3}} - (x'-y')^{\frac{2}{3}} = 2(2c)^{\frac{1}{3}}$.

21. $x = a(\cos\phi + \phi\sin\phi),\ y = a(\sin\phi - \phi\cos\phi)$.
Ans. $x'^2 + y'^2 = a^2$.

In each of the following curves, find the coördinates of the center of curvature of the curve at a point (x_0, y_0) on the curve:

22. $x^3 = ay^2$. *Ans.* $x' = -x_0 - \dfrac{9x_0^2}{2a}$,

$$y' = 4\left(x_0 + \frac{a}{3}\right)\sqrt{\frac{x_0}{a}}.$$

23. $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$. *Ans.* $x' = x_0 - \dfrac{y_0}{a}\sqrt{y_0^2 - a^2}$,

$$y' = 2y_0.$$

CHAPTER XIV

DIFFERENTIATION OF FUNCTIONS OF TWO INDEPENDENT VARIABLES

120. In the preceding chapters, the functions considered were all of one variable. The functions, however, with which we have to deal are sometimes functions of two or more independent variables. Thus, for example, the area of a rectangle is a function of two independent variables: — the two sides; and the volume of a rectangular parallelopiped is a function of three independent variables: — the three edges. In this chapter the differentiation of functions of two independent variables will be considered.

121. In a function, $f(x, y)$, of two independent variables, if real values be assigned to x and y, and the resulting value or values of $f(x, y)$ be real, the values for x, y, and $z = f(x, y)$, plotted according to the conventions of analytic geometry of three dimensions, determine a point, or points, in space. Then $z = f(x, y)$ is geometrically represented by a surface, or by surfaces.

For example, in the function $\pm\sqrt{a^2 - x^2 - y^2}$, given by the equation $z = \pm\sqrt{a^2 - x^2 - y^2}$, if real values be assigned to x and y, such that $x^2 + y^2 < a^2$, $\pm\sqrt{a^2 - x^2 - y^2}$ has real values, and the values of x, y, and $\pm\sqrt{a^2 - x^2 - y^2}$, plotted on the axes of x, y, and z according to the conventions of analytic geometry of three dimensions in rectangular coördinates, determine two points in space. Then $z = \pm\sqrt{a^2 - x^2 - y^2}$ is geometrically represented by two surfaces. As a matter of fact,

in this particular example, the two surfaces join together and give the entire surface of a sphere with center at origin and radius a.

122. Definitions. A function, $f(x, y)$, of the two independent variables x and y, is said to be **single valued** when, for values of x and y, the function has just one value.

A single-valued function, $f(x, y)$, of the two independent variables x and y is said to be **finite** when there is no set of values of x and y for which the function does not have a definite value.

A single-valued function, $f(x, y)$, of the two independent variables x and y, is said to be **continuous** for the set of values a and b, when $f(a, b)$ is finite, and $\underset{\substack{h \doteq 0 \\ k \doteq 0}}{\text{limit}}\Big[f(a+h,\ b+k)\Big] = f(a, b)$, no matter how h and k approach zero.

In what follows we shall suppose that $f(x, y)$ is single valued and continuous.

123. In a function, $f(x, y)$, of two independent variables, we have to consider:

Case I. How $f(x, y)$ varies when x varies and y remains constant;

Case II. How $f(x, y)$ varies when x remains constant and y varies;

Case III. How $f(x, y)$ varies when both x and y vary and are independent of each other.

124. It was seen above that $z = f(x, y)$ in Case III is represented geometrically by a surface. In Case I, where y is constant, $z = f(x, y)$ is represented geometrically by a plane curve formed by the intersection of the plane $y =$ the constant, with this surface. In Case II, where x is constant, $z = f(x, y)$ is represented geometrically by the plane curve formed by the intersection of the plane $x =$ the constant, with this surface.

125. In either Case I or Case II the function $f(x, y)$ is exactly like those considered in the preceding chapters, because in either case it is a function of one variable alone.

Case I. Suppose that y has the constant value y_0. Then for any value x_0 of x, $\Delta z = f(x_0 + \Delta x, y_0) - f(x_0, y_0)$.

Case II. Suppose that x has the constant value x_0. Then for any value y_0 of y, $\Delta z = f(x_0, y_0 + \Delta y) - f(x_0, y_0)$.

To distinguish between the Δz's in the above cases, we shall use the notation $\Delta_x z$ in Case I, and $\Delta_y z$ in Case II.

From the definition of a derivative, it follows that:

$\underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta_x z}{\Delta x}\right]$ is the derivative of z with respect to x when $x = x_0$ and y is constant; and

$\underset{\Delta y \doteq 0}{\text{limit}} \left[\frac{\Delta_y z}{\Delta y}\right]$ is the derivative of z with respect to y when $y = y_0$ and x is constant.

126. Definitions. $\underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta_x z}{\Delta x}\right]$ is called the **partial derivative of z with respect to x** when $x = x_0$ and y is constant. It is written $\left.\frac{\partial z}{\partial x}\right|_{x=x_0}$, and read, "partial—$d$—$x$—of—$z$ when $x = x_0$."

$\underset{\Delta y \doteq 0}{\text{limit}} \left[\frac{\Delta_y z}{\Delta y}\right]$ is called the **partial derivative of z with respect to y** when $y = y_0$ and x is constant. It is written $\left.\frac{\partial z}{\partial y}\right|_{y=y_0}$, and read, "partial—$d$—$y$—of—$z$ when $y = y_0$."

In general, for any value of x, $\underset{\Delta x \doteq 0}{\text{limit}} \left[\frac{\Delta_x z}{\Delta x}\right] = \frac{\partial z}{\partial x}$, and for any value of y, $\underset{\Delta y \doteq 0}{\text{limit}} \left[\frac{\Delta_y z}{\Delta y}\right] = \frac{\partial z}{\partial y}$.

127. To interpret $\Delta_x z$, $\left.\frac{\partial z}{\partial x}\right|_{x=x_0}$, $\Delta_y z$, and $\left.\frac{\partial z}{\partial y}\right|_{y=y_0}$ geometrically:

Suppose that y has a constant value y_0. Let A be a point, distant x_0 from the yz-plane, on the curve formed by the inter-

section of the plane $y=y_0$ with the surface $z=f(x, y)$. (See Fig. 45.) From A draw AB perpendicular to the xy-plane. On the line of intersection of the plane $y=y_0$ with the xy-plane, lay off BC equal to Δx. From C draw CD perpendicular to the xy-plane to meet the curve in D. From A draw AE parallel to BC to meet CD in E.

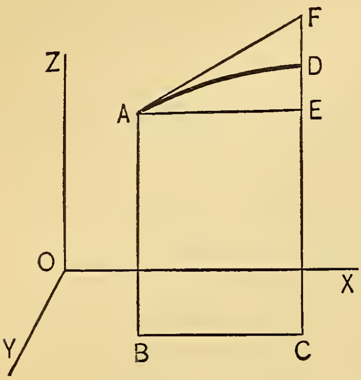


FIG. 45.

Then $\Delta_x z$ is geometrically the distance ED, and $\left.\frac{\partial z}{\partial x}\right|_{x=x_0}$ is geometrically the slope of the tangent line AF to the curve at the point A.

Suppose that x has the constant value x_0. Let A be a point on the plane curve formed by the intersection of the plane $x=x_0$ with the surface $z=f(x, y)$. (See Fig. 46.) From A draw AB perpendicular to the xy-plane. On the line of intersection of the plane $x=x_0$ with the xy-plane, lay off BG equal to Δy. From G draw GH perpendicular to the xy-plane to meet the curve in H. From A draw AK parallel to BG to meet GH in K.

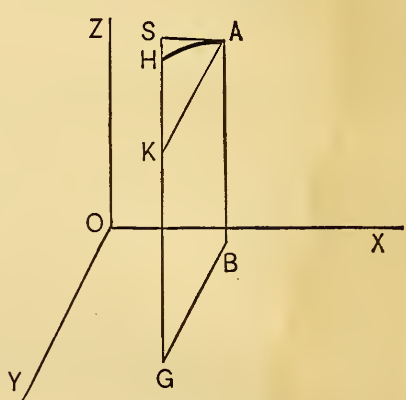


FIG. 46.

Then $\Delta_y z$ is geometrically the distance KH, and $\left.\frac{\partial z}{\partial y}\right|_{y=y_0}$ is geometrically the slope of the tangent line, AS, to the curve at the point A.

128. Let $z=f(x, y)$. Since $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are, in general, functions of x and y, we may, in general, repeat the operation of partial differentiation with respect to either x or y.

The partial derivative of $\frac{\partial z}{\partial x}$ with respect to x is $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$. It is written $\frac{\partial^2 z}{\partial x^2}$.

The partial derivative of $\frac{\partial z}{\partial y}$ with respect to y is $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$. It is written $\frac{\partial^2 z}{\partial y^2}$.

The partial derivative of $\frac{\partial z}{\partial y}$ with respect to x is $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$. It is written $\frac{\partial^2 z}{\partial x \partial y}$.

The partial derivative of $\frac{\partial z}{\partial x}$ with respect to y is $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$. It is written $\frac{\partial^2 z}{\partial y \partial x}$.

Similarly for higher derivatives. Thus $\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right)$ is the partial derivative of z, first, with respect to y; second, with respect to y; and third, with respect to x. It is written $\frac{\partial^3 z}{\partial x \partial y^2}$.

129. In Chapter XI, in the equation $y = f(x)$, where $f(x)$ is a function of one variable alone, we defined dx to be Δx, and dy to be $\frac{dy}{dx}\,dx$. Then in $z = f(x, y)$, in Case I, $dx = \Delta x$ and $dz = \frac{\partial z}{\partial x}\,dx$, and in Case II, $dy = \Delta y$, and $dz = \frac{\partial z}{\partial y}\,dy$.

To distinguish between the dz's in the above cases, we shall use the notation $d_x z$ in Case I, and $d_y z$ in Case II.

130. To interpret $d_x z$ and $d_y z$ geometrically:

Produce CD to meet AF in F. (See Fig. 45.) Then $d_x z$ is geometrically the distance EF.

Produce GH to meet AS in S. (See Fig. 46.) Then $d_y z$ is geometrically the distance KS.

131. So far we have considered differentiation only in Cases I and II. It remains to consider Case III.

In the equation $z = f(x, y)$ where x and y both vary and are independent of each other, let x have the value x_0 and y the value y_0. Then $f(x_0 + \Delta x,\ y_0 + \Delta y) - f(x_0,\ y_0)$ is the

increment produced in $f(x, y)$ by giving x and y increments. Denote it by Δz.

Then $$\Delta z = f(x_0 + \Delta x,\ y_0 + \Delta y) - f(x_0,\ y_0).$$

132. To interpret Δz geometrically:

Let A, with coördinates (x_0, y_0, z_0), be a point on the surface $z = f(x, y)$. Through A, pass planes parallel to the xz and yz planes. On the line BC of intersection of the plane ABC with the xy-plane, lay off $BC = \Delta x$. On the line BG of intersection of the plane ABG with the xy-plane, lay off $BG = \Delta y$. Through A pass a plane parallel to the xy-plane. Let ML, the line of intersection of the planes HGM and ECM, meet the plane through A parallel to the xy-plane in N and the surface in L. Then Δz is geometrically the distance NL.

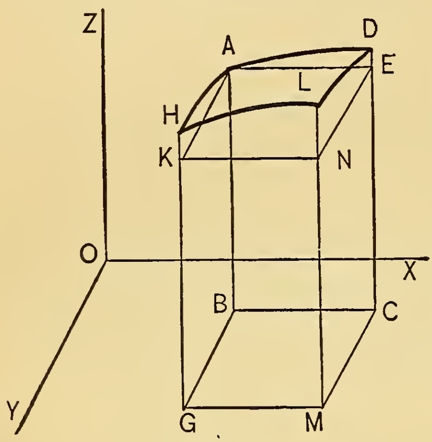


FIG. 47.

133. To define dz when x and y vary and are independent of each other:

Let A with coördinates (x_0, y_0, z_0) be a point on the surface $z = f(x, y)$. Make the same construction as in Fig. 47. (See Fig. 48.) Also, from A draw a tangent plane to the surface to meet the lines drawn perpendicular to the xy-plane in P, F, and S.

Since Δz is NL, we should wish dz to be NP. Let us define dz to be NP.

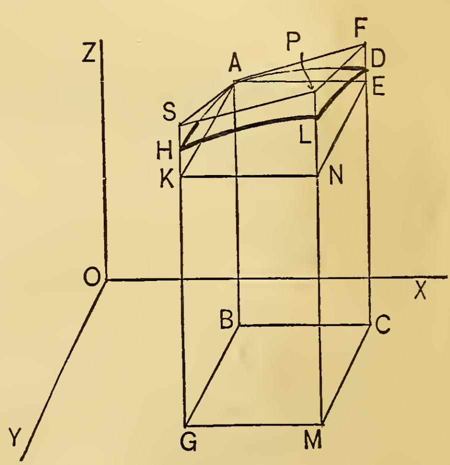


FIG. 48.

It is a theorem of solid geometry, easy of proof, that if a plane cuts a rectangular parallelopiped, the sum of two opposite edges cut off is equal to the sum of the other two opposite edges. Then since the plane $AFPS$ cuts the parallelopiped,

$$NP + \text{zero} = EF + KS.$$

Now $NP = dz$, $EF = d_x z$, and $KS = d_y z$.

$$\therefore\ dz = d_x z + d_y z.$$

From the definitions of $d_x z$ and $d_y z$, it follows that

$$dz = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy.$$

EXERCISES

In each of the three following equations, find $\Delta_x z$, $\Delta_y z$, $\frac{\partial z}{\partial x}$, and $\frac{\partial z}{\partial y}$, when $x = x_0$ and $y = y_0$:

1. $z = x^2 - 3\,xy + y^2$.

Ans. $(2\,x_0 - 3\,y_0)\Delta x + \overline{\Delta x}^2$; $(2\,y_0 - 3\,x_0)\Delta y + \overline{\Delta y}^2$; $2\,x_0 - 3\,y_0$; $2\,y_0 - 3\,x_0$.

2. $z = 3\,x^2 - 2\,y^2$.

Ans. $6\,x_0\Delta x + 3\,\overline{\Delta x}^2$; $4\,y_0\Delta y - 2\,\overline{\Delta y}^2$; $6\,x_0$; $-4\,y_0$.

3. $z = x^3 - 3\,x^2y + y^3$.

Ans. $(3\,x_0^2 - 6\,x_0y_0)\Delta x + (3\,x_0 + 3\,y_0)\overline{\Delta x}^2 + \overline{\Delta x}^3$; $(3\,y_0^2 - 3\,x_0^2)\Delta y + 3\,y_0\overline{\Delta y}^2 + \overline{\Delta y}^3$; $3\,x_0 - 6\,x_0y_0$; $3\,y_0^2 - 3\,x_0^2$.

In each of the four following equations, find $d_x z$, $d_y z$, and dz:

4. $z = \tan^{-1}\frac{y}{x}$. *Ans.* $d_x z = -\frac{y\,dx}{x^2 + y^2}$; $d_y z = \frac{x\,dy}{x^2 + y^2}$.

5. $z = \log\ \tan\frac{y}{x}$. *Ans.* $d_x z = -\frac{2\,y\,dx}{x^2 \sin\frac{2\,y}{x}}$; $d_y z = \frac{2\,dx}{x \sin\frac{2\,y}{x}}$.

6. $z = e^{xy}$. *Ans.* $d_x z = ye^{xy}dx$; $d_y z = xe^{xy}dy$.

7. $z = x^{\log y}$. *Ans.* $d_x z = x^{-1+\log y}\log y\,dx$; $d_y z = x^{\log y}\log x\,\frac{dy}{y}$.

In each of the five following equations, prove that

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y}.$$

8. $z = e^{xy}$.

9. $z = (x^2 + y^2)^{\frac{1}{2}}$.

10. $z = x \log (1 + xy)$.

11. $z = \log \tan \frac{y}{x}$.

12. $z = x^{\log y}$.

13. If $x = a \cos \theta$, $y = a \sin \theta$, prove that $\cos \theta \frac{\partial x}{\partial \theta} + \sin \theta \frac{\partial y}{\partial \theta} = 0$.

14. If $u = A \cos (x + at) + B \sin (x - at)$, prove that

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0.$$

15. If $u = \log \sqrt{(x - a)^2 + (y - b)^2}$, where $x - a$ and $y - b$ are not simultaneously zero, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

16. If $pv = a\theta$, where a is a constant, find dp in terms of dv and $d\theta$.

Ans. $dp = -\frac{p}{v} dv + \frac{p}{\theta} d\theta$.

17. If $u = \log (\tan x + \tan y)$, prove that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} = 2.$$

18. If $u = (5t^2 + 3z^2)^{\frac{1}{3}}$, prove that $u^5 \frac{\partial^2 u}{\partial z\, \partial t} + u^2 z \frac{\partial u}{\partial t} + 10zt = 0$.

19. If $u = r^n \sin n\theta$, prove that $r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$.

CHAPTER XV

INTEGRATION

134. In Chapter XI, the problems discussed were of this sort: given a function of one independent variable, to find its differential. We shall now consider the converse problem; namely, given a function of one independent variable in the form of a differential expression, that is, consisting of a function multiplied by the differential of the variable, to find the function whose differential is the given function.

135. It is evident at the outset that if we can find one function whose differential is the given function, we can find any number of such functions, because, since the differential of a constant is zero, functions which differ only in their constant terms will have the same differential. It is also true that functions which have the same differential can differ only in their constant terms, but the proof of this theorem would be beyond the scope of this book. We shall assume the truth of the statement, and on this supposition can conclude that if we can find a function whose differential is the given function, this function plus c, where c is an arbitrary constant, will contain all functions whose differential is the given function.

For example, $2\,x\,dx$ is the differential of x^2+2, x^2-5, or in general, x^2+c, where c is any constant, and x^2+c contains all functions whose differential is $2\,x\,dx$.

136. Definitions. A function whose differential is the given function is called an **indefinite integral** of the given function.

Thus, x^2+2 is an indefinite integral of $2\,x\,dx$.

The process of finding an indefinite integral of a function is called **integration.**

The constant c added to an indefinite integral is called the **arbitrary constant of integration.**

Integration is denoted by the symbol $\int$.

Thus, $\int 2x\,dx = x^2 + c.$

The symbol $\int$ is read, "indefinite integral of."

137. Integration is the inverse of differentiation. Therefore to integrate a function we must reduce it to a form where it is recognizable as the differential of some known function. The forms, called the **fundamental forms of integration**, to one or other of which we shall reduce our functions, are as follows:

I. $\int u^n\,du = \frac{u^{n+1}}{n+1} + c$, if $n \neq -1$.

II. $\int \frac{du}{u} = \log u + c.$

III. $\int a^u\,du = \frac{a^u}{\log a} + c.$

IV. $\int e^u\,du = e^u + c.$

V. $\int \cos u\,du = \sin u + c.$

VI. $\int \sin u\,du = -\cos u + c.$

VII. $\int \sec^2 u\,du = \tan u + c.$

VIII. $\int \operatorname{cosec}^2 u\,du = -\cot u + c.$

IX. $\int \sec u \tan u\,du = \sec u + c.$

X. $\int \operatorname{cosec} u \cot u\,du = -\operatorname{cosec} u + c.$

XI. $\int \tan u\,du = \log \sec u + c.$

XII. $\int \cot u\, du = \log \sin u + c.$

XIII. $\int \sec u\, du = \log(\sec u + \tan u) + c$

$$= \log \tan\left(\frac{\pi}{4} + \frac{u}{2}\right) + c.$$

XIV. $\int \operatorname{cosec} u\, du = \log(\operatorname{cosec} u - \cot u) + c = \log \tan \frac{u}{2} + c.$

XV. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + c$, or $= -\cos^{-1}\frac{u}{a} + c.$

XVI. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c$, or $= -\frac{1}{a}\cot^{-1}\frac{u}{a} + c.$

XVII. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + c$, or $= -\frac{1}{a}\operatorname{cosec}^{-1}\frac{u}{a} + c.$

XVIII. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\log\frac{u-a}{u+a} + c$, or $= \frac{1}{2a}\log\frac{a-u}{a+u} + c.$

XIX. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log(u + \sqrt{u^2 \pm a^2}) + c.$

XX. $\int \frac{du}{\sqrt{2au - u^2}} = \operatorname{vers}^{-1}\frac{u}{a} + c.$

These forms are not fundamental in the sense that no one of them can be derived from one or more of the others. They contain all the fundamental forms and others used frequently in practice.

These forms can be readily established by differentiation. For example,

$$\frac{d\left(\frac{1}{n+1}u^{n+1} + c\right)}{dx} = \frac{1}{n+1}(n+1)\, u^n\, du = u^n\, du.$$

$$\therefore \int u^n\, du = \frac{u^{n+1}}{n+1} + c.$$

138. The following theorems are of constant use in problems in integration.

Theorem I. A constant factor may be written either before or after the sign of integration.

Theorem II. The integral of an algebraic sum of two functions in the differential form is equal to the sum of the integrals of these functions.

Proof of Theorem I.

Since $d\,cu = c\,du$, where c is a constant, therefore, from the definition of an integral,

$$cu = \int c\,du. \tag{1}$$

Now $\int du = u$, by applying Form I where $n = 0$.

$$\therefore\ c\int du = cu. \tag{2}$$

Equating (1) and (2), we have

$$\int c\,du = c\int du.$$

Proof of Theorem II.

Since $d(u+v) = du + dv$, therefore, from the definition of an integral,

$$\int (du + dv) = u + v. \tag{1}$$

Now $\int du = u$ and $\int dv = v$, by Form I.

Substitute in the right-hand member of (1).

$$\therefore\ \int (du + dv) = \int du + \int dv.$$

The reasoning in Theorem II can be readily extended to any number of functions. It can therefore be shown that

$$\int (du + dv + \cdots + dw) = \int du + \int dv + \cdots + \int dw.$$

139. The following examples will illustrate how the integral of a function may be obtained in some simple cases.

EXAMPLE 1. Find $\int(x^4+x^3-3x^2+2x-4)dx.$

$$\int(x^4+x^3-3x^2+2x-4)dx=\int x^4\,dx+\int x^3\,dx+\int -3x^2\,dx$$
$$+\int 2x\,dx+\int -4\,dx,$$

by Art. 138,

$$=\int x^4\,dx+\int x^3\,dx-3\int x^2\,dx$$
$$+2\int x\,dx-4\int dx,$$

by Theorem I, Art. 138,

$$=\frac{x^5}{5}+\frac{x^4}{4}-x^3+x^2-4x+c,$$

by Form I.

EXAMPLE 2. Find $\int\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt[3]{x}}\right)dx.$

$$\int\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt[3]{x}}\right)dx=\int\frac{dx}{\sqrt{x}}+\int\frac{dx}{\sqrt[3]{x}},$$ by Theorem II, Art. 138,

$$=\int x^{-\frac{1}{2}}\,dx+\int x^{-\frac{1}{3}}\,dx$$

$$=2x^{\frac{1}{2}}+\tfrac{3}{2}x^{\frac{2}{3}}+c,$$ by Form I,

$$=2\sqrt{x}+\tfrac{3}{2}\sqrt[3]{x^2}+c.$$

EXAMPLE 3. Find $\int\frac{x^2-1}{x}\,dx.$

$$\int\frac{x^2-1}{x}\,dx=\int\left(x-\frac{1}{x}\right)dx$$

$$=\int x\,dx-\int\frac{1}{x}\,dx$$

$$=\frac{x^2}{2}-\log x+c.$$

EXERCISES

Evaluate the following integrals and in each case show by differentiation that your result is correct.

1. $\int (x^5 - 1)\, dx.$

2. $\int (x^{\frac{1}{2}} - 1)^2 dx.$

3. $\int (\sqrt{x} + \sqrt[3]{x})^2 dx.$

4. $\int (\sqrt{x^3} + 3\sqrt[3]{x^2})\, dx.$

5. $\int \left(x^{\frac{2}{3}} - \frac{a}{x^5}\right) dx.$

6. $\int \frac{x^3 - 3x^2 + 2x - 1}{x} dx.$

7. $\int \frac{x^4 - 1}{x^2} dx.$

8. $\int (a^2 - x^2)\sqrt{x}\, dx.$

9. $\int (x^2 + a^2)\sqrt[3]{x}\, dx.$

10. $\int e^{x+5}\, dx.$

11. $\int 10^{x-2}\, dx.$

12. $\int x^{n+1}\, dx.$

13. $\int \cos(x + a)\, dx.$

14. $\int \sin(x - a)\, dx.$

15. $\int \frac{dx}{x^2 - 2}.$

16. $\int \frac{dx}{x^2 + 5}.$

17. $\int \frac{dx}{x\sqrt{x^2 - 3}}.$

18. $\int \frac{dx}{\sqrt{x^2 - 3}}.$

19. $\int \frac{dx}{\sqrt{3 - x^2}}.$

20. $\int \frac{dx}{\sqrt{x^2 - 4}}.$

21. $\int \frac{dx}{\sqrt{x^2 + 4}}.$

22. $\int \frac{dx}{\sqrt{5 + x^2}}.$

23. $\int \tan^2 x\, dx.$

24. $\int \cot^2 x\, dx.$

The arbitrary constant of integration, c, can always be determined if we know the value of the integral for some particular value of the variable; for, since a constant is a number that has the same value for all values of the variable, it is known whenever its value for any value of the variable is known.

CHAPTER XVI

PLANE AREAS BY INDEFINITE INTEGRALS

RECTANGULAR COÖRDINATES

140. Let $y=f(x)$ be an equation in which $f(x)$ is single valued and continuous for all values of x between and including two values a and b. To find the area inclosed by the curve $y=f(x)$, the x-axis, and the ordinates corresponding to the abscissas a and b respectively.

In the equation $y=f(x)$, the function $f(x)$ may be positive or negative for all values of x between a and b, or it may change in sign any number of times as x increases from a to b.

141. At first suppose that $f(x)$ is positive for all values of x between a and b.

Under this supposition, the curve $y=f(x)$ is above the axis

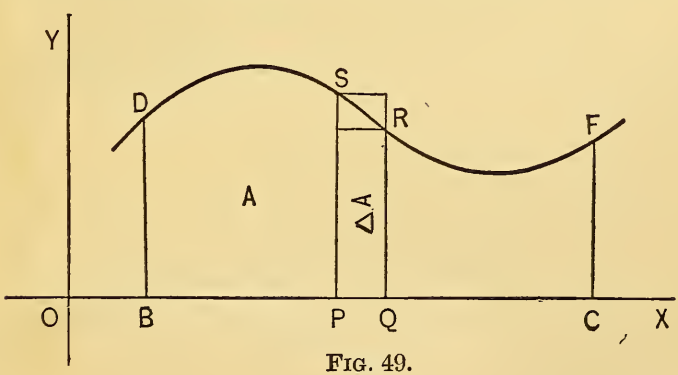


FIG. 49.

for all values of x between a and b. Suppose that it is as in Fig. 49.

Let OB and OC represent the abscissas a and b respectively.

Let OP represent an abscissa x, $a<x<b$. From B, P, and C, draw ordinates to meet the curve in D, S, and F respectively.

The area $BDSP$ is obviously a function of x, because as x assumes different values, it assumes different values. Denote it by A. Let x take an increment Δx. Then A takes an increment ΔA. In the figure, PQ represents Δx, and $PSRQ$ represents ΔA.

The areas of the rectangles SQ and RP are $f(x)\Delta x$ and $f(x+\Delta x)\Delta x$ respectively. From the figure we see therefore that

$$f(x)\Delta x<\Delta A<f(x+\Delta x)\Delta x,$$

if the curve is rising from S to R, or

$$f(x+\Delta x)\Delta x<\Delta A<f(x)\Delta x,$$

if the curve is falling from S to R. Divide by Δx.

$\therefore\ f(x)<\dfrac{\Delta A}{\Delta x}<f(x+\Delta x)$, if the curve is rising from S to R,

or, $f(x+\Delta x)<\dfrac{\Delta A}{\Delta x}<f(x)$, if the curve is falling from S to R.

As Δx approaches zero, $f(x+\Delta x)$ approaches $f(x)$ as its limit. Then, whether the curve is rising or falling from S to R, $\dfrac{\Delta A}{\Delta x}$ has a value between $f(x)$ and an expression that approaches $f(x)$ as its limit as Δx approaches zero. Therefore $\dfrac{\Delta A}{\Delta x}$ approaches $f(x)$ as its limit as Δx approaches zero. That is:

$$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta A}{\Delta x}\right]=f(x).$$

But
$$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta A}{\Delta x}\right]=\frac{dA}{dx}.$$

$$\therefore\ \frac{dA}{dx}=f(x).$$

$$\therefore\ dA=f(x)dx.$$

$$\therefore\ A=\int f(x)dx.$$

Suppose that $\int f(x)dx$ is $\phi(x)+c$. Then the area inclosed by the given curve, the x-axis, and the ordinates corresponding to the abscissas a and x is $\phi(x)+c$, where c is a constant at present undetermined. To determine c: Suppose that P moves back to B. Then x assumes the value a. Under this supposition the required area is zero, and $\phi(x)+c$ becomes $\phi(a)+c$. $\therefore\ c=-\phi(a)$.

Therefore the area inclosed by the curve, the x-axis, and the ordinates corresponding to the abscissas a and x, is $\phi(x)-\phi(a)$.

Therefore the required area $=\phi(b)-\phi(a)$, where

$$\int f(x)dx=\phi(x)+c.$$

142. Example. Find the area inclosed by the curve $y=x^2$, the x-axis, and the ordinates corresponding to the abscissas 1 and 4 respectively.

The curve is as in Fig. 50.

$$x^2\Delta x < \Delta A < (x+\Delta x)^2\Delta x.$$

$$\therefore\ x^2 < \frac{\Delta A}{\Delta x} < (x+\Delta x)^2.$$

$$\therefore\ \frac{dA}{dx}=x^2.$$

$$\therefore\ dA=x^2dx.$$

$$\therefore\ A=\int x^2dx=\frac{x^3}{3}+c.$$

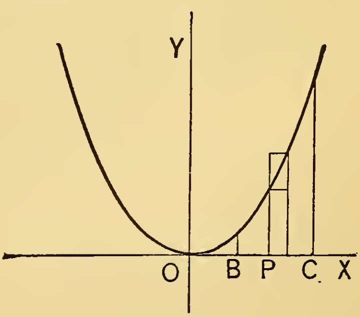


Fig. 50.

Let P move back to B. Then x becomes equal to 1, and A to zero.

$\therefore\ A=\frac{x^3}{3}+c$ becomes $0=\frac{1}{3}+c$. $\therefore\ c=-\frac{1}{3}$.

Therefore the area inclosed by the given curve, the x-axis, and the ordinates corresponding to the abscissas 1 and x respectively is

$$\frac{x^3}{3}-\frac{1}{3}.$$

Therefore the required area $=\frac{64}{3}-\frac{1}{3}=21$.

143. Let us again consider the problem stated in Art. 140, and suppose now that $f(x)$ is negative for all values of x between a and b.

Under this supposition the curve is below the x-axis for all values of x between a and b. Suppose that it is as in Fig. 51. Make the same construction as in Fig. 49, Art. 141.

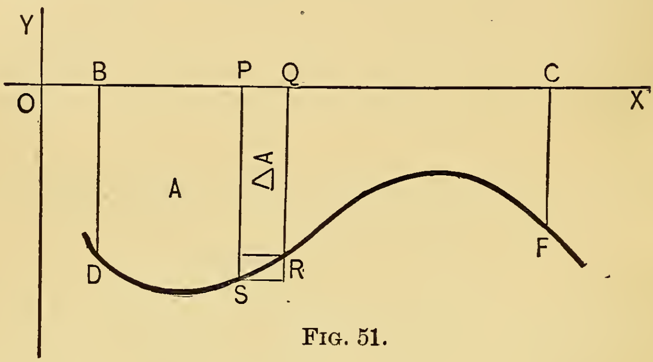


FIG. 51.

Since $f(x)$ is negative for all values of x between a and b, the areas of the rectangles RP and SQ are $-f(x+\Delta x)\,\Delta x$ and $-f(x)\,\Delta x$ respectively.

Therefore $\quad -f(x+\Delta x)\,\Delta x < \Delta A < -f(x)\,\Delta x$

if the curve is rising from S to R, or

$$-f(x)\,\Delta x < \Delta A < -f(x+\Delta x)\,\Delta x$$

if the curve is falling from S to R.

Divide by Δx. Then, whether the curve is rising or falling, $\dfrac{\Delta A}{\Delta x}$ has a value between $-f(x)$ and an expression which approaches $-f(x)$ as its limit as Δx approaches zero. Then, as in the preceding case,

$$\underset{\Delta x \doteq 0}{\text{limit}}\left[\frac{\Delta A}{\Delta x}\right] = -f(x).$$

$$\therefore\ A = -\int f(x)\,dx.$$

Therefore the required area $= -[\phi(b) - \phi(a)]$, where $\int f(x)\,dx = \phi(x) + c$.

144. Example. Find the area inclosed by the curve $y=(x-1)(x-2)$ and the x-axis.

The curve is as in Fig. 52.

The required area is the region inclosed by the part of the curve between $x=1$ and $x=2$, and the x-axis.

$$\therefore\ A=-\int(x-1)(x-2)\,dx$$

$$=-\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]+c.$$

When $x=1$, $A=0$, and therefore $c=\frac{1}{3}-\frac{3}{2}+2=\frac{5}{6}$.

$$\therefore\ A=-\tfrac{8}{3}+6-4+\tfrac{5}{6}=\tfrac{1}{6}.$$

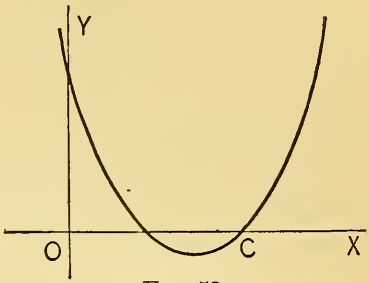


Fig. 52.

145. Suppose now that in the problem stated in Art. 140, $f(x)$ changes in sign as x increases from a to b.

Under this supposition the curve cuts the x-axis for some value or values of x between a and b.

Since there are different expressions for area according as the curve is above or below the x-axis, it follows that to find the area in this case we must find the area separately for each of the regions above the x-axis and for each below and add the results.

146. Example. Find the area inclosed by the curve $y=\sin x$, and the x-axis between $x=0$ and $x=2\pi$.

The curve cuts the x-axis when $x=\pi$.

For the region above the x-axis,

$$A=\int\sin x\,dx=-\cos x+c.$$

Therefore the area of the region above the x-axis $=2$.

For the region below the x-axis,

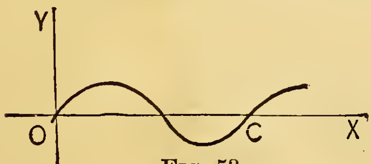


Fig. 53.

$$A=-\int\sin x\,dx=\cos x+c.$$

Therefore the area of the region below the x-axis $=2$.

Therefore the required area $=4$.

147. From Arts. 141 and 143, we see that when $f(x)$, as defined in Art. 140, does not change in sign as x increases from a to b, $\phi(b)-\phi(a)$, where $\int f(x)\,dx=\phi(x)+c$, denotes the area or the negative of the area inclosed by the curve whose equation is $y=f(x)$, the x-axis, and the ordinates corresponding to the abscissas a and b respectively. This raises the question as to what it denotes when $f(x)$ changes in sign as x increases from a to b.

Suppose that $f(x)$ changes in sign for n values of x between a and b. Let the values be c_1, c_2, c_3, $\cdots$, c_n, taken in this order as x increases from a to b.

$$\begin{aligned}\phi(b)-\phi(a) &= \phi(c_1)-\phi(a)+\phi(c_2)-\phi(c_1)+\phi(c_3)-\phi(c_2)\\ &\qquad +\cdots+\phi(b)-\phi(c_n), \text{ identically,}\\ &=[\phi(c_1)-\phi(a)]+[\phi(c_2)-\phi(c_1)]+[\phi(c_3)-\phi(c_2)]\\ &\qquad +\cdots+[\phi(b)-\phi(c_n)].\end{aligned}$$

Now each expression in square brackets denotes the area or the negative of the area of one of the regions inclosed by the curve and axis, the area being denoted when the region is above the x-axis and the negative of the area when the region is below. Therefore $\phi(b)-\phi(a)$, where $\int f(x)\,dx=\phi(x)+c$, denotes the sum of the areas of all the regions inclosed by the curve and axis above the x-axis, *minus* the sum of the areas of all the regions below.

POLAR COÖRDINATES

148. Let $r=f(\theta)$ be an equation in which $f(\theta)$ is single valued and continuous for all values of θ between and including two values α and β. To find the area inclosed by the curve $r=f(\theta)$, and the radii vectores that make angles of α and β respectively with the initial line.

149. At first suppose that $f(\theta)$ is positive for all values of θ between α and β.

Suppose that the curve $r=f(\theta)$ is as in Fig. 54.

Let OB and OC represent the radii vectores that make angles of α and β respectively with the initial line. Let OP represent a radius vector that makes an angle θ with the initial line. Let θ take an increment $\Delta\theta$. Let OS represent the radius vector that makes the angle $\theta+\Delta\theta$ with the initial line.

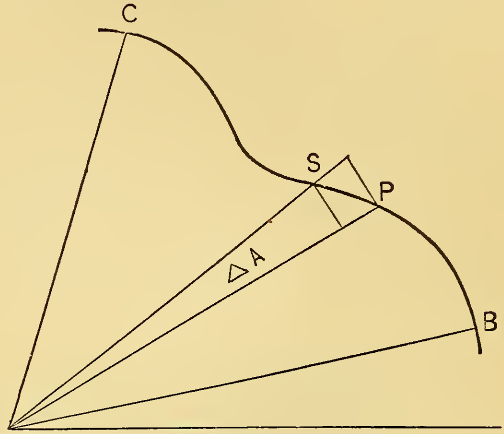


FIG. 54.

As θ takes an increment $\Delta\theta$, the area BOP, or A, takes an increment ΔA. Then from the figure we see that

$$\tfrac{1}{2}\{f(\theta)\}^2\Delta\theta < \Delta A < \tfrac{1}{2}\{f(\theta+\Delta\theta)\}^2\Delta\theta,$$

if $f(\theta)$ is increasing from P to S, or

$$\tfrac{1}{2}\{f(\theta+\Delta\theta)\}^2\Delta\theta < \Delta A < \tfrac{1}{2}\{f(\theta)\}^2\Delta\theta,$$

if $f(\theta)$ is decreasing from P to S.

Divide by $\Delta\theta$.

$$\therefore\ \tfrac{1}{2}\{f(\theta)\}^2 < \frac{\Delta A}{\Delta\theta} < \tfrac{1}{2}\{f(\theta+\Delta\theta)\}^2,$$

if $f(\theta)$ is increasing from P to S, or

$$\tfrac{1}{2}\{f(\theta+\Delta\theta)\}^2 < \frac{\Delta A}{\Delta\theta} < \tfrac{1}{2}\{f(\theta)\}^2,$$

if $f(\theta)$ is decreasing from P to S.

As $\Delta\theta$ approaches zero, $\tfrac{1}{2}\{f(\theta+\Delta\theta)\}^2$ approaches $\tfrac{1}{2}\{f(\theta)\}^2$ as its limit. Then, whether $f(\theta)$ is increasing or decreasing

from P to S, $\frac{\Delta A}{\Delta\theta}$ has a value between $\frac{1}{2}\{f(\theta)\}^2$ and an expression that approaches $\frac{1}{2}\{f(\theta)\}^2$ as a limit as $\Delta\theta$ approaches zero.

That is,
$$\lim_{\Delta\theta \doteq 0}\left[\frac{\Delta A}{\Delta\theta}\right] = \tfrac{1}{2}\{f(\theta)\}^2.$$

$$\therefore\ \frac{dA}{d\theta} = \tfrac{1}{2}\{f(\theta)\}^2.$$

$$\therefore\ A = \tfrac{1}{2}\int\{f(\theta)\}^2 d\theta$$

$$= \phi(\beta) - \phi(\alpha),$$

where
$$\tfrac{1}{2}\int\{f(\theta)\}^2 d\theta = \phi(\theta) + c.$$

150. Next, suppose that in the problem stated in Art. 148 $f(\theta)$ is negative for all values of θ between α and β, or changes in sign as θ increases from α to β.

Since $f(\theta)$ is real, $\frac{1}{2}\{f(\theta)\}^2$ is positive for all values of θ between α and β. Then, whether $f(\theta)$ is positive or negative, $\frac{dA}{d\theta}$ is always positive.

$$\therefore\ A = \tfrac{1}{2}\int\{f(\theta)\}^2 d\theta$$

$$= \phi(\beta) - \phi(\alpha),$$

where
$$\tfrac{1}{2}\int\{f(\theta)\}^2 d\theta = \phi(\theta) + c.$$

151. Example. Find the area inclosed by the curve $r = a\sqrt{1-\cos\theta}$.

$$A = \frac{a^2}{2}\int(1-\cos\theta)\,d\theta$$

$$= \frac{a^2}{2}\Big[\theta - \sin\theta\Big] + c.$$

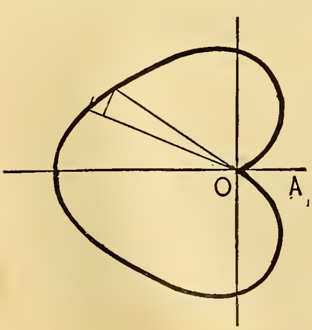


Fig. 55.

When $\theta = 0$, $A = 0$. $\therefore\ c = 0$.

When $\theta = 2\pi$, $A = \frac{a^2}{2}\Big[2\pi - 0\Big] = \pi a^2$.

Therefore the required area $= \pi a^2$.

EXERCISES

When tables are necessary, use them to four places.

1. Find the area inclosed by the parabola $y^2 = 8x$ and the double ordinate corresponding to the abscissa 2. *Ans.* $\frac{32}{3}$.

2. Find the area inclosed by the curve $y = \frac{x^2 - 1}{x}$, the x-axis, and the ordinates corresponding to the abscissas $\frac{1}{2}$ and 1 respectively. *Ans.* 0.318.

3. Find the area inclosed by the curve $y = \frac{1}{x^2 - 4}$, the x-axis, and the ordinates corresponding to the abscissas -1 and $+1$ respectively. *Ans.* 0.549.

4. Show that the area cut off from a parabola by any double ordinate is $\frac{2}{3}$ the area of the circumscribing rectangle.

5. Find the area inclosed by the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coördinate axes. *Ans.* $\frac{a^2}{6}$.

6. Find the area inclosed by one arch of the curve $y = \cos\left(x + \frac{\pi}{4}\right)$, and the x-axis. *Ans.* 2.

7. Find the area inclosed by the curve $y = \frac{1}{\sqrt{4 - x^2}}$, the x-axis, and the ordinates corresponding to the abscissas -2 and $+2$ respectively. *Ans.* π.

8. Find the area inclosed by the semicubical parabola $ay^2 = x^3$, the y-axis, and the line $y = b$. *Ans.* $\frac{3b}{5}\sqrt[3]{ab^2}$.

In each of the seven following curves, find the area inclosed by the curve, the x-axis, and the ordinates corresponding to the abscissas set opposite the equation.

9. $y = x(x-1)(x-2)$. $x = -\frac{1}{2}$, $x = 3$. *Ans.* 3.141.

10. $y = \frac{(x-1)(x-2)}{x}$. $x = \frac{1}{2}$, $x = 1$. *Ans.* 0.261.

11. $xy = 4$. $x = 1$, $x = 5$. *Ans.* 6.44.

12. $y = \sin x$. $x = 1$, $x = 3$. *Ans.* 1.530.

13. $y = \cos x$. $x = 2,\ x = 7$. *Ans.* 3.566.

14. $y = \dfrac{1}{\sqrt{2 - x^2}}$. $x = 1,\ x = 1\frac{1}{3}$. *Ans.* 0.446.

15. $y = 10^x$. $x = \frac{1}{2},\ x = 2$. *Ans.* 42.06.

In each of the four following equations, find the area inclosed by the curve, and the radii vectores corresponding to the values of θ set opposite the equation.

16. $r = \tan\theta$. $\theta = 0,\ \theta = \dfrac{\pi}{4}$. *Ans.* 0.107.

17. $r = \sin\frac{1}{2}\theta + \cos\frac{1}{2}\theta$. $\theta = 0,\ \theta = \dfrac{\pi}{4}$. *Ans.* 0.539.

18. $r = a\theta$. $\theta = 35^\circ,\ \theta = 70^\circ$. *Ans.* $0.266\, a^2$.

19. $r = e^{\frac{1}{2}\theta}$. $\theta = \dfrac{\pi}{4},\ \theta = \dfrac{\pi}{2}$. *Ans.* 1.308.

Find the area inclosed by each of the following curves:

20. $r^2 = a^2(1 - \cos\theta)$. *Ans.* $(\pi + 2)\, a^2$.

21. $r = a(1 - \cos\theta)$. *Ans.* $\frac{3}{2}\pi a^2$.

Make use of $\int \cos^2\theta\, d\theta = \dfrac{\theta}{2} + \dfrac{1}{2}\sin\theta\cos\theta$.

22. $r = a\sin\theta$. *Ans.* $\dfrac{\pi}{4}a^2$.

23. $r = 1 + 2\cos\theta$. (Outer curve.) *Ans.* $2\pi + \frac{3}{2}\sqrt{3}$.

24. $r = 2 + \cos\theta$. *Ans.* $\frac{9}{2}\pi$.

25. $r = a + b\cos\theta,\ |b| > |a|$. (Outer curve.)

Ans.

$\left(a^2 + \dfrac{b^2}{2}\right)\theta + \dfrac{3\,|a|}{2}\sqrt{b^2 - a^2}$, where $\theta = \cos^{-1} - \dfrac{a}{b} = \sin^{-1}\dfrac{\sqrt{b^2 - a^2}}{|b|}$, if a and b have the same sign; $\left(a^2 + \dfrac{b^2}{2}\right)\left(\pi - \theta\right) + \dfrac{3\,|a|}{2}\sqrt{b^2 - a^2}$, where $\theta = \cos^{-1} - \dfrac{a}{b} = \sin^{-1}\dfrac{\sqrt{b^2 - a^2}}{|b|}$, if a and b have opposite signs.

26. $r = a + b\cos\theta,\ |a| > |b|$. *Ans.* $(2\,a^2 + b^2)\dfrac{\pi}{2}$.

27. $r = a + b\cos\theta,\ |a| = |b|$. (See 21, above.) *Ans* $\frac{3}{2}\pi a^2$.

CHAPTER XVII

METHODS OF INTEGRATION

In Chapter XV, we saw how to obtain the indefinite integral of a differential expression in a few simple cases. In this chapter, we shall investigate methods whereby the integral of more complicated expressions can be found.

For convenience in writing, the arbitrary constant will be omitted in this and the following chapter.

INTEGRATION BY SUBSTITUTION

152. In many problems in integration the integral can be evaluated by reducing it to one of the fundamental forms by means of the substitution of a new variable. The following examples will illustrate the process.

Example 1. Find $\int \frac{dx}{x-1}$.

The differential of the denominator is the numerator. The integral will therefore reduce to Form II by the substitution of u for $x-1$.

Let $x-1=u$. $\therefore\ dx=du$, and $\int \frac{dx}{x-1}=\int \frac{du}{u}$.

Now $$\int \frac{du}{u}=\log u, \text{ by Form II.}$$

$$\therefore \int \frac{dx}{x-1}=\log(x-1).$$

Example 2. Find $\int \frac{dx}{\sqrt{1+x}}$.

The differential of the expression under the radical in the denominator is the numerator. The integral will therefore reduce to Form I by the substitution of u for $1+x$.

Let $1+x=u$. $\therefore\ dx=du$, and $\int\frac{dx}{\sqrt{1+x}}=\int\frac{du}{\sqrt{u}}=\int u^{-\frac{1}{2}}\,du.$

Now $$\int u^{-\frac{1}{2}}\,du=2\,u^{\frac{1}{2}},\text{ by Form I.}$$

$$\therefore\ \int\frac{dx}{\sqrt{1+x}}=2(1+x)^{\frac{1}{2}}.$$

EXAMPLE 3. Find $\int\sec^2(2-3\,x)dx.$

This integral appears as if it may reduce to Form VII.

Let $$2-3\,x=u.$$

$$\therefore\ dx=-\frac{du}{3},\text{ and }\int\sec^2(2-3\,x)dx=-\tfrac{1}{3}\int\sec^2 u\,du.$$

Now $$-\tfrac{1}{3}\int\sec^2 u\,du=-\tfrac{1}{3}\tan u,\text{ by Form VII.}$$

$$\therefore\ \int\sec^2(2-3\,x)dx=-\tfrac{1}{3}\tan(2-3\,x).$$

EXAMPLE 4. Find $\int e^{a+bx}dx.$

This integral appears as if it may reduce to Form IV.

Let $a+bx=u$. $\therefore\ dx=\frac{du}{b}$, and $\int e^{a+bx}\,dx=\frac{1}{b}\int e^u\,du.$

Now $$\frac{1}{b}\int e^u\,du=\frac{1}{b}e^u,\text{ by Form IV.}$$

$$\therefore\ \int e^{a+bx}\,dx=\frac{1}{b}e^{a+bx}.$$

EXAMPLE 5. Find $\int\frac{dx}{ax^2+bx+c}.$

The denominator is rational and of the second degree. The integral should therefore reduce to Form XVI or XVIII. Divide both numerator and denominator by a.

$$\therefore\int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\int\frac{dx}{x^2+\frac{b}{a}x+\frac{c}{a}}.$$

Now $$x^2+\frac{b}{a}x+\frac{c}{a}\equiv x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}$$

$$\equiv\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2},$$

or $$\equiv\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}.$$

$$\therefore \int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\int\frac{dx}{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}},$$

or $$=\frac{1}{a}\int\frac{dx}{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}}.$$

Let $$x+\frac{b}{2a}=u. \quad \therefore\ dx=du,$$

and $$\int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\int\frac{du}{u^2+\frac{4ac-b^2}{4a^2}},$$

or $$=\frac{1}{a}\int\frac{du}{u^2-\frac{b^2-4ac}{4a^2}}.$$

Suppose that b^2-4ac is negative.

Then $4ac-b^2$ is positive, $\sqrt{4ac-b^2}$ is real, and

$$\int\frac{du}{u^2+\frac{4ac-b^2}{4a^2}}=\frac{2a}{\sqrt{4ac-b^2}}\tan^{-1}\frac{u}{\frac{\sqrt{4ac-b^2}}{2a}}, \text{ by Form XVI.}$$

$$\therefore \int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\,\frac{2a}{\sqrt{4ac-b^2}}\tan^{-1}\frac{x+\frac{b}{2a}}{\frac{\sqrt{4ac-b^2}}{2a}}$$

$$=\frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}},$$

if b^2-4ac is negative.

Suppose that $b^2 - 4ac$ is positive.

Then $\sqrt{b^2 - 4ac}$ is real, and

$$\int \frac{du}{u^2 + \frac{b^2 - 4ac}{4a^2}} = \frac{1}{a} \frac{2a}{2\sqrt{b^2-4ac}} \log \frac{u - \frac{\sqrt{b^2 - 4ac}}{2a}}{u + \frac{\sqrt{b^2 - 4ac}}{2a}}, \text{ by Form XVIII.}$$

$$\therefore \int \frac{dx}{ax^2 + bx + c} = \frac{2a}{2\sqrt{b^2 - 4ac}} \log \frac{\left(x + \frac{b}{2a}\right) - \frac{\sqrt{b^2 - 4ac}}{2a}}{\left(x + \frac{b}{2a}\right) + \frac{\sqrt{b^2 - 4ac}}{2a}}$$

$$= \frac{1}{\sqrt{b^2 - 4ac}} \log \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}},$$

if $b^2 - 4ac$ is positive.

Suppose that $b^2 - 4ac = 0$.

Then either $\frac{1}{a}\int \frac{du}{u^2 + \frac{4ac - b^2}{4a^2}}$ or $\frac{1}{a}\int \frac{du}{u^2 - \frac{b^2 - 4ac}{4a^2}}$ becomes $\frac{1}{a}\int \frac{du}{u^2}$, which, by Form I, reduces to $-\frac{1}{a}\frac{1}{u}$.

$$\therefore \int \frac{dx}{ax^2 + bx + c} = -\frac{1}{a} \frac{1}{x + \frac{b}{2a}}$$

$$= -\frac{2}{2ax + b}, \text{ if } b^2 - 4ac = 0.$$

EXERCISES

Show by integration that:

1. $\displaystyle\int \frac{dx}{16x^2 - 9} = \frac{1}{24} \log \frac{4x - 3}{4x + 3}.$

2. $\displaystyle\int \frac{dx}{9x^2 + 16} = \frac{1}{12} \tan^{-1} \frac{3x}{4}.$

3. $\displaystyle\int \frac{dx}{\sqrt{1 + 16x^2}} = \frac{1}{4} \log (4x + \sqrt{1 + 16x^2}).$

4. $\int \frac{dx}{\sqrt{1-16x^2}} = \frac{1}{4}\sin^{-1}4x.$

5. $\int \frac{dx}{\sqrt{8x-x^2}} = \text{vers}^{-1}\frac{x}{4}.$

6. $\int (2+3x)^2 dx = \frac{1}{9}(2+3x)^3.$

7. $\int \frac{dx}{2+5x} = \frac{1}{5}\log(2+5x).$

8. $\int \frac{x\,dx}{5+2x} = \frac{1}{2}x - \frac{5}{4}\log(5+2x).$

9. $\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3}.$

10. $\int x\sqrt{a+bx}\,dx = -\frac{2(2a-3bx)\sqrt{(a+bx)^3}}{15b^2}.$

11. $\int \frac{dx}{\sqrt{3-4x}} = -\frac{\sqrt{3-4x}}{2}.$

12. $\int (e^x+e^{-x})^2 dx = \frac{1}{2}(e^{2x}+4x-e^{-2x}).$

13. $\int (e^{ax}+e^{-ax})\,dx = \frac{1}{a}(e^{ax}-e^{-ax}).$

14. $\int \frac{e^{5x}dx}{e^x-1} = \frac{1}{4}e^{4x}+\frac{1}{3}e^{3x}+\frac{1}{2}e^{2x}+e^x+\log(e^x-1).$

15. $\int (\sin 3x+\cos 3x)\,dx = \frac{1}{3}(-\cos 3x+\sin 3x).$

16. $\int \frac{3x-1}{x^2-9}\,dx = \log(x+3)^{\frac{5}{3}}(x-3)^{\frac{4}{3}}.$

17. $\int \frac{3x-1}{x^2+9}\,dx = \frac{3}{2}\log(x^2+9) - \frac{1}{3}\tan^{-1}\frac{x}{3}.$

18. $\int \frac{dx}{\sqrt{2x^2+3x-3}} = \frac{1}{\sqrt{2}}\log\left(\sqrt{2x^2+3x-3}+x\sqrt{2}+\frac{3}{2\sqrt{2}}\right).$

19. $\int \frac{dx}{\sqrt{3+3x-2x^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\frac{4x-3}{\sqrt{33}}.$

20. $$\int \frac{dx}{2x^2+3x-4} = \frac{1}{\sqrt{41}} \log \frac{4x+3-\sqrt{41}}{4x+3+\sqrt{41}}.$$

21. $$\int \frac{dx}{3x^2-2x+4} = \frac{1}{\sqrt{11}} \tan^{-1} \frac{3x-1}{\sqrt{11}}.$$

22. $$\int \frac{dx}{x^2+2x+3} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{x+1}{\sqrt{2}}.$$

23. $$\int \frac{dx}{\sqrt{2x^2+3x+1}} = \frac{1}{\sqrt{2}} \log (4x+3+2\sqrt{2}\sqrt{2x^2+3x+1}).$$

24. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}} \log (2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}),$$
if $a > 0$.
$$= \frac{1}{\sqrt{-a}} \sin^{-1} \frac{-2ax-b}{\sqrt{b^2-4ac}}, \quad \text{if } a < 0.$$

25. $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a}\int \frac{dx}{\sqrt{ax^2+bx+c}}.$$

SUGGESTION. $\dfrac{x}{\sqrt{ax^2+bx+c}} \equiv \dfrac{1}{2a} \dfrac{2ax+b}{\sqrt{ax^2+bx+c}} - \dfrac{b}{2a} \dfrac{1}{\sqrt{ax^2+bx+c}}$, identically.

26. $$\int \frac{3x-1}{x^2+x-6}\,dx = \log [(x+3)^2(x-2)].$$

27. $$\int \frac{x\,dx}{3-4x} = -\frac{1}{4}x - \frac{3}{16} \log (3-4x).$$

28. $$\int \frac{x^2\,dx}{3x+2} = \frac{1}{6}x^2 - \frac{2}{9}x + \frac{4}{27}\log (3x+2).$$

29. $$\int \frac{x\,dx}{\sqrt{2-7x}} = -\frac{2}{49}[2\sqrt{2-7x} - \frac{1}{3}\sqrt{(2-7x)^3}].$$

30. $$\int \frac{x\,dx}{\sqrt{3+2x}} = \frac{1}{4}\left[\frac{2}{3}\sqrt{(3+2x)^3} - 6\sqrt{(3+2x)}\right].$$

31. $$\int \frac{x^2\,dx}{\sqrt{1+x}} = \frac{2}{5}\sqrt{(1+x)^5} - \frac{4}{3}\sqrt{(1+x)^3} + 2\sqrt{1+x}.$$

32. $$\int \frac{dx}{a+bx} = \frac{1}{b} \log (a+bx).$$

33. $\int \frac{x\,dx}{x^4-1} = \frac{1}{4}\log\frac{x^2-1}{x^2+1}.$

34. $\int \frac{dx}{x\sqrt{2-5x}} = \frac{1}{\sqrt{2}}\log\frac{\sqrt{2-5x}-\sqrt{2}}{\sqrt{2-5x}+\sqrt{2}}.$

35. $\int \frac{dx}{x\sqrt{5x-2}} = \sqrt{2}\tan^{-1}\sqrt{\frac{5x-2}{2}}.$

36. $\int \frac{dx}{x\sqrt{a+bx}} = \frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right)$, if $a>0$.

$$= \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}}, \text{ if } a<0.$$

37. $\int \frac{dx}{x\sqrt{x^2-3x+7}} = -\frac{1}{\sqrt{7}}\log\left(\frac{\sqrt{x^2-3x+7}+\sqrt{7}}{x}-\frac{3}{2\sqrt{7}}\right).$

SUGGESTION. Let $x=\frac{1}{u}$.

38. $\int \frac{dx}{x\sqrt{4x^2+3x-1}} = \sin^{-1}\left(\frac{3x-2}{5x}\right).$

39. $\int \frac{dx}{x\sqrt{ax^2+bx+c}} = -\frac{1}{\sqrt{c}}\log\left(\frac{\sqrt{ax^2+bx+c}+\sqrt{c}}{x}+\frac{b}{2\sqrt{c}}\right)$, if $c>0$.

$$= \frac{1}{\sqrt{-c}}\sin^{-1}\left(\frac{bx+2c}{x\sqrt{b^2-4ac}}\right), \text{ if } c<0$$

40. $\int \frac{dx}{\sqrt{a+bx}} = \frac{2\sqrt{a+bx}}{b}.$

41. $\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a}\log(ax^2+bx+c) - \frac{b}{2a}\int\frac{dx}{ax^2+bx+c}.$

42. $\int \frac{dx}{x^{\frac{1}{2}}-x^{\frac{1}{3}}} = 6\left[\frac{x^{\frac{1}{2}}}{3}+\frac{x^{\frac{1}{3}}}{2}+x^{\frac{1}{6}}+\log(x^{\frac{1}{6}}-1)\right].$

SUGGESTION. Let $x^{\frac{1}{6}}=u$.

43. $\int \frac{dx}{x^{\frac{5}{8}}+x^{\frac{3}{4}}} = 8\left[\frac{x^{\frac{1}{4}}}{2}-x^{\frac{1}{8}}+\log(x^{\frac{1}{8}}+1)\right].$

44. $\displaystyle\int \frac{x\,dx}{x^{\frac{1}{2}}+x^{\frac{1}{4}}} = 4\left[\frac{x^{\frac{3}{2}}}{6} - \frac{x^{\frac{5}{4}}}{5} + \frac{x}{4} - \frac{x^{\frac{3}{4}}}{3} + \frac{x^{\frac{1}{2}}}{2} - x^{\frac{1}{4}} + \log\left(x^{\frac{1}{4}}+1\right)\right].$

45. $\displaystyle\int \frac{x^{\frac{3}{2}}\,dx}{2a-x} = -2\left[\frac{x^{\frac{3}{2}}}{3} + 2ax^{\frac{1}{2}} + \sqrt{2}\,a^{\frac{3}{2}}\log\frac{x^{\frac{1}{2}}-\sqrt{2a}}{x^{\frac{1}{2}}+\sqrt{2a}}\right].$

46. $\displaystyle\int \frac{dx}{2+5\sin x} = \frac{1}{\sqrt{21}}\log\frac{2\tan\frac{x}{2}+5-\sqrt{21}}{2\tan\frac{x}{2}+5+\sqrt{21}}.$

SUGGESTION. $\displaystyle\int \frac{dx}{2+5\sin x} = \int \frac{dx}{2\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+10\sin\frac{x}{2}\cos\frac{x}{2}}$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{2\tan^2\frac{x}{2}+10\tan\frac{x}{2}+2}. \quad \text{Let } \tan\frac{x}{2} = u.$$

47. $\displaystyle\int \frac{dx}{2-5\cos x} = \frac{1}{\sqrt{21}}\log\frac{\sqrt{7}\tan\frac{x}{2}-\sqrt{3}}{\sqrt{7}\tan\frac{x}{2}+\sqrt{3}}.$

48. $\displaystyle\int \frac{dx}{5+2\sin x} = \frac{2}{\sqrt{21}}\tan^{-1}\frac{5\tan\frac{x}{2}+2}{\sqrt{21}}.$

49. $\displaystyle\int \frac{dx}{5-2\cos x} = \frac{2}{\sqrt{21}}\tan^{-1}\left(\sqrt{\frac{7}{3}}\tan\frac{x}{2}\right).$

RATIONAL FRACTIONS

153. A rational fraction in x multiplied by dx can be integrated, whenever the denominator can be factored into linear or quadratic factors, by breaking the fraction up into partial fractions.

EXAMPLE. Find $\displaystyle\int \frac{dx}{(x-1)^2(x-2)}.$

Let $$\frac{1}{(x-1)^2(x-2)} \equiv \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x-2}.$$

Clear of fractions.

$$\therefore\ 1 \equiv A(x-2) + B(x-1)(x-2) + C(x-1)^2.$$

Equate coefficients.

$$\therefore\ B + C = 0.$$
$$A - 3B - 2C = 0.$$
$$-2A + 2B + C = 1.$$

By solving the equations, we get $A = -1$, $B = -1$, $C = 1$.

$$\therefore\ \frac{1}{(x-1)^2(x-2)} \equiv -\frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{1}{x-2}.$$

$$\therefore\ \int \frac{dx}{(x-1)^2(x-2)} = -\int \frac{dx}{(x-1)^2} - \int \frac{dx}{x-1} + \int \frac{dx}{x-2}$$
$$= \frac{1}{x-1} + \log \frac{x-2}{x-1}.$$

EXERCISES

Show by integration that:

1. $\displaystyle\int \frac{2x-1}{(x-1)(x-2)}\,dx = \log \frac{(x-2)^3}{x-1}.$

2. $\displaystyle\int \frac{x\,dx}{(x+1)(x+3)(x+5)} = \frac{1}{8} \log \frac{(x+3)^6}{(x+5)^5(x+1)}.$

3. $\displaystyle\int \frac{2x^3+1}{(x+1)(x+2)}\,dx = x^2 - 6x + \log \frac{(x+2)^{15}}{x+1}.$

4. $\displaystyle\int \frac{dx}{x(x^2+1)} = \log \frac{x}{\sqrt{x^2+1}}.$

5. $\displaystyle\int \frac{x^2-3}{(x+2)(x^2+1)}\,dx = \frac{1}{5} \log (x+2)(x^2+1)^2 - \frac{8}{5} \tan^{-1} x.$

6. $\displaystyle\int \frac{x^2+x}{(x-1)^2(x^2+4)}\,dx = -\frac{2}{5(x-1)} + \frac{11}{50} \log \frac{(x-1)^2}{x^2+4} + \frac{2}{25} \tan^{-1} \frac{x}{2}.$

7. $\displaystyle\int \frac{x\,dx}{(x+1)(x^2+2x+3)} = \frac{1}{4} \log \frac{x^2+2x+3}{(x+1)^2} + \frac{1}{\sqrt{2}} \tan^{-1} \frac{x+1}{\sqrt{2}}.$

8. $\displaystyle\int \frac{dx}{x^3+1} = \frac{1}{6} \log \frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{2}}.$

INTEGRATION BY PARTS

154. In Art. 99 we saw that

$$d(uv) = u\,dv + v\,du.$$

Transpose. $\qquad \therefore\ u\,dv = d(uv) - v\,du.$

$$\therefore\ \int u\,dv = \int d(uv) - \int v\,du.$$

Now $\qquad \int d(uv) = uv.$

$$\therefore\ \int u\,dv = uv - \int v\,du. \tag{1}$$

The use of (1) is called **integration by parts.** By its aid, the desired integral can often be found.

EXAMPLE 1. Find $\int \sin^2 x\,dx.$

Let $\qquad u = \sin x$, and $dv = \sin x\,dx.$

$$\therefore\ du = \cos x\,dx, \text{ and } v = -\cos x.$$

Substitute in (1).

$$\therefore\ \int \sin^2 x\,dx = -\sin x \cos x + \int \cos^2 x\,dx. \tag{2}$$

Now $\qquad \cos^2 x = 1 - \sin^2 x.$

$$\therefore\ \int \cos^2 x\,dx = \int dx - \int \sin^2 x\,dx$$

$$= x - \int \sin^2 x\,dx.$$

Substitute in (2).

$$\therefore\ \sin^2 x\,dx = -\sin x \cos x + x - \int \sin^2 x\,dx.$$

Transpose $\int \sin^2 x\,dx$, and add.

$$\therefore\ 2\int \sin^2 x\,dx = -\sin x \cos x + x.$$

$$\therefore\ \int \sin^2 x\,dx = -\frac{1}{2}\sin x \cos x + \frac{x}{2}.$$

EXAMPLE 2. Find $\int e^{ax} \sin nx\, dx$.

Let $$u = e^{ax}, \text{ and } dv = \sin nx\, dx.$$

$$\therefore\ du = ae^{ax}\, dx, \text{ and } v = -\frac{1}{n} \cos nx.$$

Substitute in (1).

$$\therefore\ \int e^{ax} \sin nx\, dx = -\frac{1}{n} e^{ax} \cos nx + \frac{a}{n} \int e^{ax} \cos nx\, dx. \qquad (2)$$

In $\int e^{ax} \cos nx\, dx$, let $u = e^{ax}$, and $dv = \cos nx\, dx$.

$$\therefore\ du = ae^{ax}, \text{ and } v = \frac{1}{n} \sin nx.$$

Substitute in (1).

$$\therefore\ \int e^{ax} \cos nx\, dx = \frac{1}{n} e^{ax} \sin nx - \frac{a}{n} \int e^{ax} \sin nx\, dx. \qquad (3)$$

Substitute (3) in (2).

$$\therefore\ \int e^{ax} \sin nx\, dx = -\frac{1}{n} e^{ax} \cos nx + \frac{a}{n}\left[\frac{1}{n} e^{ax} \sin nx - \frac{a}{n} \int e^{ax} \sin nx\, dx\right]$$

$$= \frac{ae^{ax} \sin nx - ne^{ax} \cos nx}{n^2} - \frac{a^2}{n^2} \int e^{ax} \sin nx\, dx.$$

Transpose $-\frac{a^2}{n^2} \int e^{ax} \sin nx\, dx$, and add.

$$\therefore\ \left(1 + \frac{a^2}{n^2}\right) \int e^{ax} \sin nx\, dx = \frac{ae^{ax} \sin nx - ne^{ax} \cos nx}{n^2}.$$

$$\therefore\ \int e^{ax} \sin nx\, dx = \frac{ae^{ax} \sin nx - ne^{ax} \cos nx}{a^2 + n^2}.$$

EXERCISES

Show by integration that:

1. $\int x \log x\, dx = \frac{x^2}{2}\left[\log x - \frac{1}{2}\right]$.

2. $\int x^2 \log x\, dx = \frac{x^3}{3}\left[\log x - \frac{1}{3}\right]$.

3. $\int x^m \log x\, dx = \frac{x^{m+1}}{m+1}\left[\log x - \frac{1}{m+1}\right]$.

4. $\int x \sin x \, dx = \sin x - x \cos x.$

5. $\int x^2 \sin x \, dx = 2x \sin x - (x^2 - 2) \cos x.$

6. $\int x^2 \cos x \, dx = 2x \cos x + (x^2 - 2) \sin x.$

7. $\int xe^{ax} \, dx = \frac{e^{ax}}{a^2}(ax - 1).$

8. $\int x^2 e^{ax} \, dx = e^{ax}\left[\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right].$

9. $\int \sin^{-1} x \, dx = x \sin^{-1} x + \sqrt{1 - x^2}.$

10. $\int \cos^{-1} x \, dx = x \cos^{-1} x - \sqrt{1 - x^2}.$

11. $\int \tan^{-1} x \, dx = x \tan^{-1} x - \frac{1}{2} \log (1 + x^2).$

12. $\int \log x \, dx = x \log x - x.$

13. $\int \cos^2 x \, dx = \frac{1}{2} \sin x \cos x + \frac{1}{2} x.$

14. $\int \sin x \cos x \, dx = \frac{1}{2} \sin^2 x.$

15. $\int \sin^n x \cos x \, dx = \frac{\sin^{n+1} x}{n + 1}.$

16. $\int \sin x \cos^n x \, dx = -\frac{\cos^{n+1} x}{n + 1}.$

17. $\int \sin nx \cos mx \, dx = -\frac{\cos (n - m)x}{2(n - m)} - \frac{\cos (n + m)x}{2(n + m)}.$

18. $\int \sin nx \sin mx \, dx = \frac{\sin (n - m)x}{2(n - m)} - \frac{\sin (n + m)x}{2(n + m)}.$

19. $\int \cos nx \cos mx \, dx = \frac{\sin (n - m)x}{2(n - m)} + \frac{\sin (n + m)x}{2(n + m)}.$

20. $\int e^{ax} \cos nx \, dx = \frac{ae^{ax} \cos nx + ne^{ax} \sin nx}{a^2 + n^2}.$

CHAPTER XVIII

REDUCTION FORMULAS

ALGEBRAIC REDUCTION FORMULAS

155. These are formulas whereby the integral

$$\int x^m (a + bx^n)^p \, dx$$

may be made to depend upon a similar integral with a more convenient value for m or p, or, in some cases, may be completely evaluated. They are:

$$\int x^m(a+bx^n)^p\,dx$$
$$= \frac{x^{m-n+1}(a+bx^n)^{p+1}}{(np+m+1)b} - \frac{(m-n+1)a}{(np+m+1)b}\int x^{m-n}(a+bx^n)^p\,dx, \quad (A)$$

$$\int x^m(a+bx^n)^p\,dx$$
$$= \frac{x^{m+1}(a+bx^n)^p}{np+m+1} + \frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}\,dx, \quad (B)$$

$$\int x^m(a+bx^n)^p\,dx$$
$$= \frac{x^{m+1}(a+bx^n)^{p+1}}{(m+1)a} - \frac{(np+m+n+1)b}{(m+1)a}\int x^{m+n}(a+bx^n)^p\,dx, \quad (C)$$

$$\int x^m(a+bx^n)^p\,dx$$
$$= -\frac{x^{m+1}(a+bx^n)^{p+1}}{n(p+1)a} + \frac{np+m+n+1}{(p+1)a}\int x^m(a+bx^n)^{p+1}\,dx, \quad (D)$$

Formula (A) is used to reduce the exponent m, (B) to reduce the exponent p, (C) to increase the exponent m, and (D) to increase the exponent p. Formula (C) or (D) is used when m or p is negative.

156. Derivation of Formulas.

To derive (*A*):

Integrate by parts where $x^{m-n+1}=u$.

$$u=x^{m-n+1}, \text{ and } \qquad dv=(a+bx^n)^p x^{n-1}dx.$$

$$\therefore\ du=(m-n+1)x^{m-n}, \text{ and } v=\frac{(a+bx^n)^{p+1}}{nb(p+1)}.$$

$$\therefore \int x^m(a+bx^n)^p dx$$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{nb(p+1)}-\frac{m-n+1}{nb(p+1)}\int x^{m-n}(a+bx^n)^{p+1}dx. \quad (1)$$

Now

$$\int x^{m-n}(a+bx^n)^{p+1}dx=\int x^{m-n}(a+bx^n)(a+bx^n)^p dx$$

$$=a\int x^{m-n}(a+bx^n)^p dx+b\int x^m(a+bx^n)^p dx.$$

Substitute in (1), transpose, add, and divide by $\dfrac{np+m+1}{n(p+1)}$.

$$\therefore \int x^m(a+bx^n)^p dx$$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{(np+m+1)b}-\frac{(m-n+1)a}{(np+m+1)b}\int x^{m-n}(a+bx^n)^p dx,$$

which is formula (*A*).

To derive (*C*):

Transpose the integrals in (*A*) and divide by $\dfrac{(m-n+1)a}{(np+m+1)b}$.

$$\therefore \int x^{m-n}(a+bx^n)^p dx$$

$$=\frac{x^{m-n+1}(a+bx^n)^{p+1}}{(m-n+1)a}-\frac{(np+m+1)b}{(m-n+1)a}\int x^m(a+bx^n)^p dx.$$

Let $m-n=m'$, or $m=n+m'$, substitute and drop the accent.

$$\therefore \int x^m(a+bx^n)^p dx$$

$$=\frac{x^{m+1}(a+bx^n)^{p+1}}{(m+1)a}-\frac{(np+m+n+1)b}{(m+1)a}\int x^{m+n}(a+bx^n)^p dx,$$

which is formula (*C*).

To derive (*B*):

Integrate by parts where $(a+bx^n)^p = u$.

$$u = (a+bx^n)^p, \text{ and } \qquad dv = x^m\,dx.$$

$$\therefore\ du = bnpx^{n-1}(a+bx^n)^{p-1}dx, \text{ and } v = \frac{x^{m+1}}{m+1}.$$

$$\therefore \int x^m(a+bx^n)^p\,dx$$

$$= \frac{x^{m+1}(a+bx^n)^p}{m+1} - \frac{bnp}{m+1}\int x^{m+n}(a+bx^n)^{p-1}dx. \qquad (1)$$

Now

$$\int x^m(a+bx^n)^p\,dx = \int x^m(a+bx^n)\,(a+bx^n)^{p-1}dx$$

$$= a\int x^m(a+bx^n)^{p-1}dx + b\int x^{m+n}(a+bx^n)^{p-1}dx.$$

$$\therefore \int x^{m+n}(a+bx^n)^{p-1}dx = \frac{1}{b}\int x^m(a+bx^n)^p\,dx - \frac{a}{b}\int x^m(a+bx^n)^{p-1}dx.$$

Substitute in (1), transpose, add, and divide by $\dfrac{np+m+1}{m+1}$.

$$\therefore \int x^m(a+bx^n)^p\,dx$$

$$= \frac{x^{m+1}(a+bx^n)^p}{np+m+1} + \frac{anp}{np+m+1}\int x^m(a+bx^n)^{p-1}dx,$$

which is formula (*B*).

To derive (*D*):

Transpose the integrals in (*B*) and divide by $-\dfrac{anp}{np+m+1}$.

$$\therefore \int x^m(a+bx^n)^{p-1}dx$$

$$= -\frac{x^{m+1}(a+bx^n)^p}{anp} + \frac{np+m+1}{anp}\int x^m(a+bx^n)^p\,dx.$$

Let $p-1=p'$ or $p=1+p'$, substitute and drop the accent.

$$\therefore \int x^m(a+bx^n)^p\,dx$$

$$= -\frac{x^{m+1}(a+bx^n)^{p+1}}{n(p+1)a} + \frac{np+m+n+1}{n(p+1)a}\int x^m(a+bx^n)^{p+1}dx,$$

which is formula (*D*).

These formulas fail when a denominator becomes zero.

157. As an illustration of the method of application of these formulas, consider the following example.

Example. Find $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}$.

By an application of formula (A) this integral may be reduced to an expression containing $\int \frac{dx}{\sqrt{a^2-x^2}}$, and can therefore be integrated.

$$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = \int x^2(a^2-x^2)^{-\frac{1}{2}}\,dx.$$

Apply formula (A). $m=2$, $n=2$, $p=-\frac{1}{2}$, $a=a^2$, $b=-1$.

$$\therefore \int x^2(a^2-x^2)^{-\frac{1}{2}}\,dx = -\frac{x(a^2-x^2)^{\frac{1}{2}}}{2} + \frac{a^2}{2}\int \frac{dx}{\sqrt{a^2-x^2}}$$

$$= -\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

$$\therefore \int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

EXERCISES

Show by integration that:

1. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

2. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right).$

3. $\int \frac{dx}{(a^2-x^2)^{\frac{5}{2}}} = \frac{x}{3\,a^2(a^2-x^2)^{\frac{3}{2}}}\cdot\frac{3\,a^2-2\,x^2}{a^2}.$

4. $\int x\sqrt{a^2-x^2}\cdot dx = -\frac{1}{3}\sqrt{(a^2-x^2)^3}.$

5. $\int x^2\sqrt{a^2+x^2}\,dx = \frac{x}{8}(a^2+2\,x^2)\sqrt{a^2+x^2} - \frac{a^4}{8}\log\left(x+\sqrt{a^2+x^2}\right).$

6. $\int x^2\sqrt{a^2-x^2}\,dx = -\frac{x}{4}\sqrt{(a^2-x^2)^3} + \frac{a^2}{8}\left(x\sqrt{a^2-x^2}+a^2\sin^{-1}\frac{x}{a}\right).$

7. $\displaystyle\int\frac{x^2\,dx}{\sqrt{x^2\pm a^2}}=\frac{x}{2}\sqrt{x^2\pm a^2}\mp\frac{a^2}{2}\log\left(x+\sqrt{x^2\pm a^2}\right).$

8. $\displaystyle\int\frac{dx}{x^2\sqrt{a^2-x^2}}=-\frac{\sqrt{a^2-x^2}}{a^2x}.$

9. $\displaystyle\int(a^2-x^2)^{\frac{3}{2}}\,dx=\frac{x}{8}(5\,a^2-2\,x^2)\sqrt{a^2-x^2}+\frac{3\,a^4}{8}\sin^{-1}\frac{x}{a}.$

10. $\displaystyle\int\frac{x\,dx}{\sqrt{2\,ax-x^2}}=-\sqrt{2\,ax-x^2}+a\,\text{vers}^{-1}\frac{x}{a}.$

11. $\displaystyle\int\frac{dx}{x\sqrt{2\,ax-x^2}}=-\frac{\sqrt{2\,ax-x^2}}{ax}.$

12. $$\int\frac{x^2\,dx}{\sqrt{ax^2+bx+c}}=\left(\frac{x}{2\,a}-\frac{3\,b}{4\,a^2}\right)\sqrt{ax^2+bx+c}$$
$$+\frac{3\,b^2-4\,ac}{8\,a^2}\int\frac{dx}{\sqrt{ax^2+bx+c}}.$$

SUGGESTION. $ax^2+bx+c\equiv a\left[\left(x+\dfrac{b}{2\,a}\right)^2-\dfrac{b^2-4\,ac}{4\,a^2}\right].$ Let $u=x+\dfrac{b}{2\,a}.$

TRIGONOMETRIC REDUCTION FORMULAS

158. The following are reduction formulas for trigonometric functions:

$$\int\tan^n x\,dx=\frac{\tan^{n-1}x}{n-1}-\int\tan^{n-2}x\,dx. \tag{1}$$

$$\int\cot^n x\,dx=-\frac{\cot^{n-1}x}{n-1}-\int\cot^{n-2}x\,dx. \tag{2}$$

$$\int\sin^m x\cos^n x\,dx=-\frac{\sin^{m-1}x\cos^{n+1}x}{m+n}+\frac{m-1}{m+n}\int\sin^{m-2}x\cos^n x\,dx. \tag{3}$$

$$\int\sin^m x\cos^n x\,dx=\frac{\sin^{m+1}x\cos^{n-1}x}{m+n}+\frac{n-1}{m+n}\int\sin^m x\cos^{n-2}x\,dx \tag{4}$$

$$\int\frac{\cos^n x\,dx}{\sin^m x}=-\frac{\cos^{n+1}x}{(m-1)\sin^{m-1}x}-\frac{n-m+2}{m-1}\int\frac{\cos^n x\,dx}{\sin^{m-2}x}. \tag{5}$$

$$\int\frac{\cos^n x\,dx}{\sin^m x}=\frac{\cos^{n-1}x}{(n-m)\sin^{m-1}x}+\frac{n-1}{n-m}\int\frac{\cos^{n-2}x\,dx}{\sin^m x}. \tag{6}$$

$$\int \sin^m x\, dx = -\frac{\sin^{m-1} x \cos x}{m} + \frac{m-1}{m}\int \sin^{m-2} x\, dx. \qquad (7)$$

$$\int \cos^n x\, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\, dx. \qquad (8)$$

$$\int \frac{dx}{\sin^m x} = -\frac{\cos x}{(m-1)\sin^{m-1} x} + \frac{m-2}{m-1}\int \frac{dx}{\sin^{m-2} x}. \qquad (9)$$

$$\int \frac{dx}{\cos^n x} = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}. \qquad (10)$$

159. Derivation of some of formulas.

To derive (1):

$$\begin{aligned}\int \tan^n x\, dx &= \int \tan^{n-2} x \tan^2 x\, dx\\ &= \int \tan^{n-2} x\,(\sec^2 x - 1)\, dx\\ &= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx\\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx.\end{aligned}$$

To derive (3):

Integrate by parts where $\sin^{m-1} x = u$.

$$u = \sin^{m-1} x, \text{ and } dv = \sin x \cos^n x\, dx.$$

$$\therefore\ du = (m-1)\cos^{m-2} x, \text{ and } v = -\frac{\cos^{n+1} x}{n+1}.$$

$$\therefore \int \sin^m x \cos^n x\, dx$$

$$= -\frac{\sin^{m-1} x \cos^{n+1} x}{n+1} + \frac{m-1}{n+1}\int \sin^{m-2} x \cos^{n-2} x\, dx.$$

Now

$$\begin{aligned}\int \sin^{m-2} x \cos^{n+2} x\, dx &= \int \sin^{m-2} x \cos^n x \cos^2 x\, dx\\ &= \int \sin^{m-2} x \cos^n x\,(1 - \sin^2 x)\, dx\\ &= \int \sin^{m-2} x \cos^n x\, dx - \int \sin^m x \cos^n x\, dx.\end{aligned}$$

Substitute, transpose, add, and divide by $\dfrac{m+n}{n+1}$.

$$\therefore \int \sin^m x \cos^n x\, dx = -\frac{\sin^{m-1}x\cos^{n+1}x}{m+n} + \frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\, dx.$$

To derive (5):

Substitute $m-2=-m'$, or $m=2-m'$, in (3), transpose, divide by $\dfrac{m'-1}{n-m'+2}$, and omit the accent.

The derivations of the others are left as exercises to the student.

EXERCISES

Show by integration that:

1. $\displaystyle\int \tan^5 x\, dx = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \log\sec x.$

2. $\displaystyle\int \cot^3 x\, dx = -\frac{\operatorname{cosec}^2 x}{2} - \log\sin x.$

3. $\displaystyle\int \sin^2 x\cos^4 x\, dx = \left(\frac{\cos x}{8} + \frac{\cos^3 x}{12} - \frac{\cos^5 x}{3}\right)\frac{\sin x}{2} + \frac{x}{16}.$

4. $\displaystyle\int \frac{\cos^4 x\, dx}{\sin^3 x} = -\frac{\cos^3 x}{2\sin^2 x} - \frac{3}{2}\cos x - \frac{3}{2}\log\tan\frac{x}{2}.$

5. $\displaystyle\int \frac{\sin^2 x\, dx}{\cos^5 x} = \frac{\sin x}{4\cos^4 x} - \frac{\sin x}{8\cos^2 x} - \frac{1}{8}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$

6. $\displaystyle\int \frac{dx}{\sin^6 x} = -\frac{\cos x}{5\sin^5 x} - \frac{4\cos x}{15\sin^3 x} - \frac{8}{15}\cot x.$

7. $\displaystyle\int \frac{dx}{\sin^4 x\cos^3 x} = -\frac{1}{3\sin^3 x\cos^2 x} - \frac{5}{3\sin x\cos^2 x} + \frac{5\sin x}{2\cos^2 x}$
$$+\frac{5}{2}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

8. $\displaystyle\int \sec^6 x\, dx = \tan x\left(\frac{1}{5\cos^4 x} + \frac{4}{15\cos^2 x} + \frac{8}{15}\right).$

9. $\displaystyle\int \operatorname{cosec}^5 x\, dx = -\cot x\left(\frac{1}{4\sin^3 x} + \frac{3}{8\sin x}\right) + \frac{3}{8}\log\tan\frac{x}{2}.$

10. $\displaystyle\int \tan^3 x\sec^5 x\, dx = \frac{\sec^7 x}{7} - \frac{\sec^5 x}{5}.$

CHAPTER XIX

SUMMATION OF $f(x)\Delta x$. PLANE AREAS IN RECTANGULAR COÖRDINATES

160. Let $y=f(x)$ be an equation in which $f(x)$ is single valued and continuous for all values of x between and including two values a and b. Let $\dfrac{b-a}{n}=\Delta x$, where n is any positive integer.

Definition. The symbol $\sum\limits_{x=a}^{x=b} f(x)\Delta x$ is used to denote

$$f(a)\Delta x+f(a+\Delta x)\Delta x+f(a+2\,\Delta x)\Delta x+\cdots+f(a+(n-1)\Delta x)\Delta x$$

for any value of n a positive integer.

161. To interpret $\sum\limits_{x=a}^{x=b} f(x)\Delta x$ geometrically:

At first, suppose that $f(x)$ is positive for all values of x between a and b.

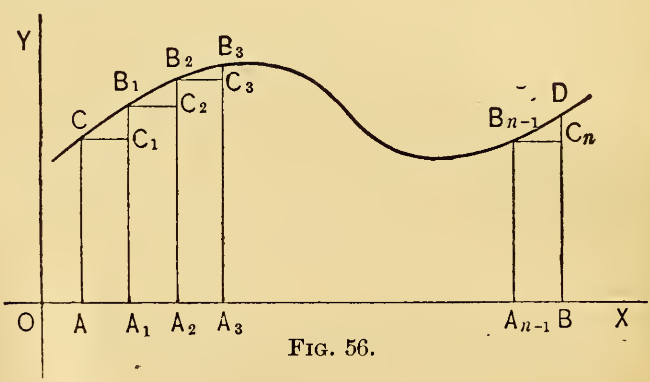


FIG. 56.

Under this supposition the curve $y=f(x)$ is above the x-axis for all values of x between a and b. Suppose that it is as in Fig. 56.

Let OA and OB represent the abscissas a and b respectively.

Divide AB into n equal parts. Each part is of length $\frac{b-a}{n}$. Call each part Δx. Denote the successive points of division of AB by $A_1, A_2, A_3, \cdots, A_{n-1}$. At $A, A_1, A_2, A_3, \cdots, A_{n-1}, B$, erect ordinates to meet the curve in $C, B_1, B_2, B_3, \cdots, B_{n-1}, D$ respectively. From $C, B_1, B_2, \cdots, B_{n-1}$, draw lines parallel to the x-axis to meet the next succeeding ordinates in $C_1, C_2, C_3, \cdots, C_n$ respectively.

The symbol $\sum\limits_{x=a}^{x=b} f(x)\Delta x$ denotes the sum of the areas of the rectangles, $CA_1, B_1A_2, B_2A_3, \cdots, B_{n-1}B$.

Next, suppose that $f(x)$ is negative for all values of x between a and b.

Under this supposition, the curve $y=f(x)$ is below the x-axis for all values of x between a and b. Suppose that it is as in Fig. 57.

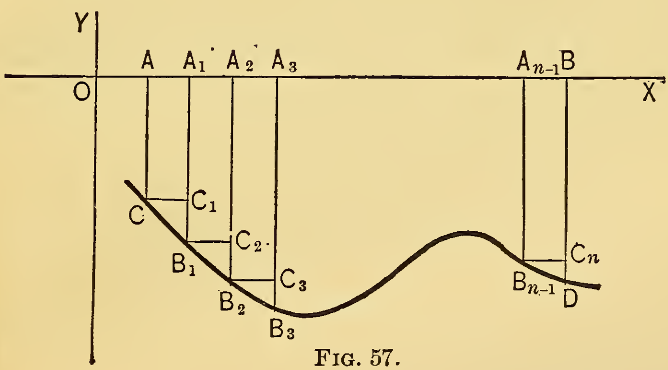


FIG. 57.

Make the same construction as in Fig. 56.

Since $f(x)$ is negative for all values of x between a and b, each term of $\sum\limits_{x=a}^{x=b} f(x)\Delta x$ is negative. Then $-\sum\limits_{x=a}^{x=b} f(x)\Delta x$ denotes the sum of the areas of the rectangles $CA_1, B_1A_2, B_2A_3, \cdots, B_{n-1}B$, and therefore $\sum\limits_{x=a}^{x=b} f(x)\Delta x$ denotes the negative of the sum of the areas of these rectangles.

Next, suppose that $f(x)$ changes in sign as x increases from a to b.

Under this supposition, the curve $y=f(x)$ crosses the x-axis for some value or values of x between a and b.

Suppose that it is as in Fig. 58.

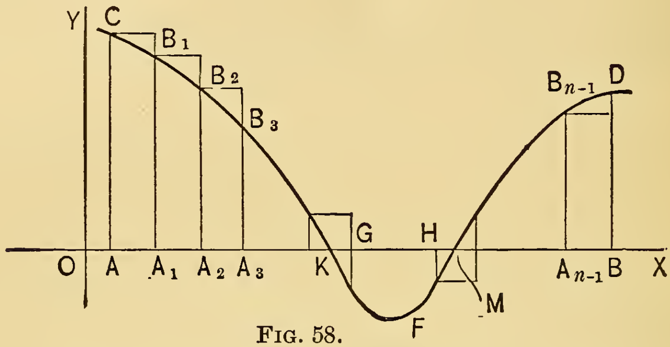


FIG. 58.

Make the same construction as in Fig. 56 or Fig. 57.

From the preceding investigation, it is evident that in this case, $\sum_{x=a}^{x=b} f(x)\Delta x$ denotes the sum of the areas of certain rectangles, *minus* the sum of the areas of others. Thus, in Fig. 58, it denotes the sum of the areas of the rectangles from A to G, *minus* the sum of the areas of the rectangles from G to H, *plus* the sum of the areas of the rectangles from H to B, where G and H are the first and last points of division, respectively, of the line AB, when the curve is below the x-axis.

162. Definitions. Those of the rectangles CA_1, B_1A_2, B_2A_3, $\cdots$, $B_{n-1}B$ of the preceding article whose bases not on the x-axis lie wholly between the curve and the x-axis are called **inscribed rectangles.**

Thus, CA_1, B_1A_2, B_2A_3 of Fig. 56, or CA_1, B_1A_2, B_2A_3 of Fig. 57, are inscribed rectangles.

Those of the rectangles CA_1, B_1A_2, B_2A_3, $\cdots$, $B_{n-1}B$ of the preceding article such that the curve lies wholly between the base not on the x-axis and the x-axis are called **circumscribed rectangles.**

Thus, CA_1, B_1A_2, B_2A_3 of Fig. 58 are circumscribed rectangles.

EXERCISES

In each of the following problems plot the curve, draw the rectangles, and calculate:

1. $\sum_{x=1}^{x=2} x^2 \Delta x$, (*a*) when $n=5$; (*b*) when $n=10$.
Ans. (*a*) 2.04; (*b*) 2.19.

2. $\sum_{x=0}^{x=2} \sin x \Delta x$, when $n=4$. *Ans.* 1.16.

3. $\sum_{x=1}^{x=2} \log_{10} x \Delta x$, when $n=5$. *Ans.* 0.137.

In Chapter XVI we regarded area as the inverse of a differential. It is more convenient, however, to regard it as the limit of the sum of a set of rectangles described as explained in Art. 161.

163. Let $y=f(x)$ be an equation in which $f(x)$ is single valued and continuous for all values of x between and including two values a and b. To investigate the area between the curve $y=f(x)$, the x-axis, and the ordinates corresponding to the abscissas a and b respectively, from the point of view of the limit of the sum of the areas of a set of rectangles.

164. At first suppose that $f(x)$ is positive for all values of x between a and b.

Suppose that the curve is as in Fig. 59. Let OA and OB represent the abscissas a and b respectively. Divide AB into

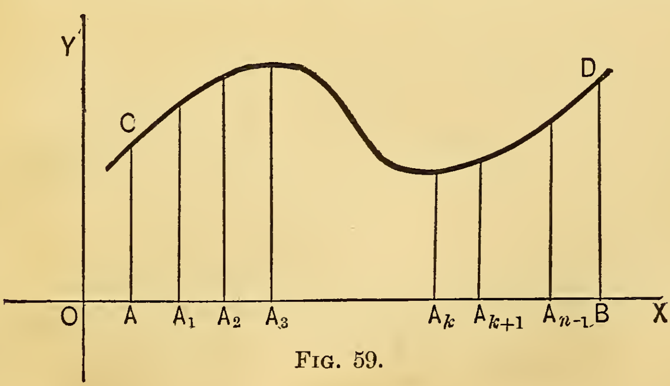


Fig. 59.

n equal parts. Call each part Δx. Denote the successive points of division of AB by $A_1, A_2, A_3, \cdots, A_{n-1}$ respectively. At $A_1, A_2, A_3, \cdots, A_{n-1}$, erect ordinates to the curve. Let $x_1, x_2, x_3, \cdots, x_{n-1}$, denote the abscissas $OA_1, OA_2, OA_3, \cdots, OA_{n-1}$ respectively.

The ordinates drawn at $A_1, A_2, A_3, \cdots, A_{n-1}$, divide the area up into infinitesimal strips of area. Let $\Delta_k A$ represent the strip whose base is $A_k A_{k+1}$. Then, by Art. 141,

$$\operatorname*{limit}_{\Delta x \doteq 0}\left[\frac{\Delta_k A}{\Delta x}\right] = f(x_k).$$

Therefore, from the definition of a limit,

$$\frac{\Delta_k A}{\Delta x} = f(x_k) + \epsilon_k,$$

where ϵ_k is infinitesimal as $\Delta x \doteq 0$, or when $n = \infty$.

$$\therefore\ \Delta_k A = f(x_k)\,\Delta x + \epsilon_k \Delta x.$$

Let k take in succession the values $1, 2, 3, \cdots, n-1$. Let $\Delta_k A$ and ϵ_k when $x = a$ be denoted by $\Delta_0 A$ and ϵ_0 respectively.

Then
$$\Delta_0 A = f(a)\,\Delta x + \epsilon_0 \Delta x,$$
$$\Delta_1 A = f(x_1)\,\Delta x + \epsilon_1 \Delta x,$$
$$\Delta_2 A = f(x_2)\,\Delta x + \epsilon_2 \Delta x,$$
$$\cdots\cdots\cdots\cdots$$
$$\Delta_{n-1} A = f(x_{n-1})\,\Delta x + \epsilon_{n-1} \Delta x.$$

Therefore
$$\begin{aligned} A &= \Delta_0 A + \Delta_1 A + \Delta_2 A + \cdots + \Delta_{n-1} A \\ &= f(a)\,\Delta x + f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_{n-1})\Delta x \\ &\qquad + (\epsilon_0 + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1})\,\Delta x \\ &= f(a)\,\Delta x + f(a + \Delta x)\,\Delta x + f(a + 2\,\Delta x)\,\Delta x + \cdots \\ &\qquad + f(a + (n-1)\,\Delta x)\,\Delta x \\ &\qquad + (\epsilon_0 + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1})\,\Delta x \\ &= \sum_{x=a}^{x=b} f(x)\,\Delta x + (\epsilon_0 + \epsilon_1 + \epsilon_2 + \cdots + \epsilon_{n-1})\,\Delta x. \end{aligned}$$

To show that $\underset{n=\infty}{\text{limit}}\left[(\epsilon_0+\epsilon_1+\epsilon_2+\cdots+\epsilon_{n-1})\,\Delta x\right]=0$:

$$|(\epsilon_0+\epsilon_1+\epsilon_2+\cdots+\epsilon_{n-1})| \leqq |\epsilon_0|+|\epsilon_1|+|\epsilon_2|+\cdots+|\epsilon_{n-1}|.$$

Of the infinitesimals $|\epsilon_0|$, $|\epsilon_1|$, $|\epsilon_2|$, $\cdots$, $|\epsilon_{n-1}|$, let $|\epsilon_i|$ be one which is not less than any of the others.

$$\therefore\ |(\epsilon_0+\epsilon_1+\epsilon_2+\cdots+\epsilon_{n-1})| \leqq n\,|\epsilon_i|.$$

$$\therefore\ |(\epsilon_0+\epsilon_1+\epsilon_2+\cdots+\epsilon_{n-1})\,\Delta x| \leqq n\,|\epsilon_i|\,\Delta x.$$

Now $n\,\Delta x = b-a$.

$$\therefore\ |(\epsilon_0+\epsilon_1+\epsilon_2+\cdots+\epsilon_{n-1})\,\Delta x| \leqq (b-a)\,|\epsilon_i|.$$

When $n=\infty$, $\epsilon_i \doteq 0$ and therefore $(b-a)\,|\epsilon_i| \doteq 0$.

$$\therefore\ \underset{n=\infty}{\text{limit}}\left[|(\epsilon_0+\epsilon_1+\epsilon_2+\cdots+\epsilon_{n-1})\,\Delta x|\right]=0.$$

$$\therefore\ \underset{n=\infty}{\text{limit}}\left[(\epsilon_0+\epsilon_1+\epsilon_2+\cdots+\epsilon_{n-1})\,\Delta x\right]=0.$$

$$\therefore\ A=\underset{n=\infty}{\text{limit}}\sum_{x=a}^{x=b} f(x)\Delta x.$$

165. Next, suppose that $f(x)$, in Art. 163, is negative for all values of x between a and b.

Suppose that the curve is as in Fig. 60.

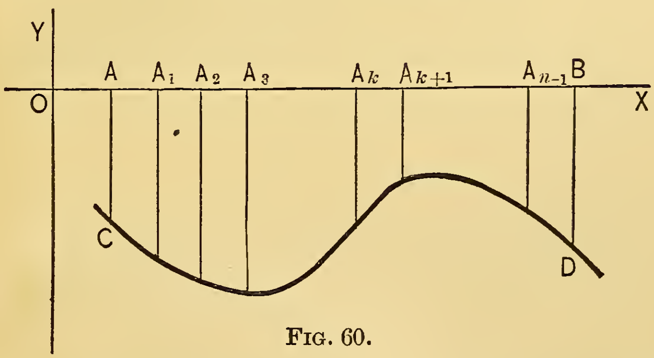


FIG. 60.

Make the same construction as in Fig. 59, Art. 164.

Let x_1, x_2, x_3, $\cdots$, x_{n-1}, denote the abscissas OA_1, OA_2, OA_3, $\cdots$, OA_{n-1} respectively.

Then, as in Art. 143,

$$\lim_{\Delta x \doteq 0}\left[\frac{\Delta_k A}{\Delta x}\right] = -f(x_k).$$

$$\therefore \quad \Delta_k A = -f(x_k)\Delta x + \epsilon_k \Delta x.$$

By reasoning exactly similar to that employed in Art. 164, we find in this case that:

$$A = -\lim_{n=\infty}\sum_{x=a}^{x=b} f(x)\,\Delta x.$$

166. To evaluate (*a*) $\lim\limits_{n=\infty}\sum\limits_{x=a}^{x=b} f(x)\Delta x$ of Art. 164, and (*b*) $-\lim\limits_{n=\infty}\sum\limits_{x=a}^{x=b} f(x)\Delta x$ of Art. 165:

(*a*) By Art. 141, the area inclosed by the curve $y=f(x)$, the x-axis, and the ordinates corresponding to the abscissas a and b respectively is $\phi(b)-\phi(a)$, where $\int f(x)\,dx = \phi(x)+c$. By the preceding article, this area also $= \lim\limits_{n=\infty}\sum\limits_{x=a}^{x=b} f(x)\Delta x$. Therefore $\lim\limits_{n=\infty}\sum\limits_{x=a}^{x=b} f(x)\Delta x = \phi(b)-\phi(a)$, where $\int f(x)\,dx = \phi(x)+c$.

(*b*) By Art. 143, the area inclosed by the curve $y=f(x)$, the x-axis, and the ordinates corresponding to the abscissas a and b respectively is $-[\phi(b)-\phi(a)]$, where $\int f(x)\,dx = \phi(x)+c$. By the preceding article, this area also $= -\lim\limits_{n=\infty}\sum\limits_{x=a}^{x=b} f(x)\,\Delta x$.

Therefore $-\lim\limits_{n=\infty}\sum\limits_{x=a}^{x=b} f(x)\Delta x = -[\phi(b)-\phi(a)]$, where $\int f(x)\,dx = \phi(x)+c$.

167. As illustrations of the method of application of the above principles to problems in area, consider the following examples:

EXAMPLE 1. Find the area inclosed by the curve $4y = x^3 - 6x^2 + 11x + 4$, the x-axis, and the ordinates corresponding to the abscissas 0 and 4 respectively.

The curve is as in Fig. 61.

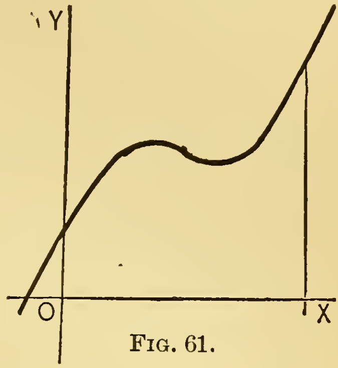


FIG. 61.

The required area

$$= \underset{n=\infty}{\text{limit}} \sum_{x=0}^{x=4} \tfrac{1}{4}(x^3 - 6x^2 + 11x + 4)\Delta x$$

$$= \frac{1}{4}\left(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} + 4x\right)_{x=4} - \frac{1}{4}\left(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} + 4x\right)_{x=0}$$

$= 10.$

EXAMPLE 2. Find the area inclosed by the curve $4y = x^3 - 6x^2 + 11x - 12$, the x-axis, and the ordinates corresponding to the abscissas 0 and 4 respectively.

The curve is as in Fig. 62.

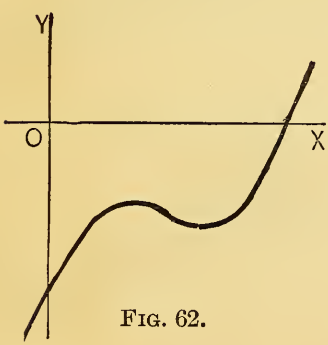


FIG. 62.

The required area

$$= -\underset{n=\infty}{\text{limit}} \sum_{x=0}^{x=4} \tfrac{1}{4}(x^3 - 6x^2 + 11x - 12)\Delta x$$

$$= \frac{1}{4}\left(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 12x\right)_{x=4} - \frac{1}{4}\left(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 12x\right)_{x=0}$$

$= 6.$

168. Next, suppose that $f(x)$ in Art. 163 changes sign as x increases from a to b. The curve $y = f(x)$ therefore crosses the x-axis for some value or values of x between a and b. Since there are different expressions for area according as the curve is above or below the x-axis, we must, in this case, find the area for each of the regions above the x-axis, and for each below, and add the results.

169. Example. Find the area inclosed by the curve $4y = x^3 - 6x^2 + 11x - 6$, the x-axis, and the ordinates corresponding to the abscissas 0 and 4 respectively.

The curve is as in Fig. 63.

Denote the points where it crosses the x-axis by A, C, and D. Let B be the foot of the ordinate whose abscissa is 4.

Divide OA, AC, CD, and DB into n_1, n_2, n_3, and n_4 equal parts respectively. Denote these equal parts by $\Delta_1 x$, $\Delta_2 x$, $\Delta_3 x$, $\Delta_4 x$ respectively.

$$\text{The required area} = -\underset{n_1=\infty}{\text{limit}} \sum_{x=0}^{x=1} \tfrac{1}{4}(x^3 - 6x^2 + 11x - 6)\Delta_1 x$$

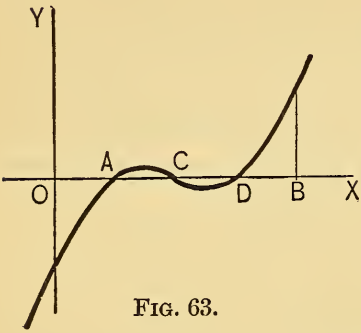


Fig. 63.

$$+\underset{n_2=\infty}{\text{limit}} \sum_{x=1}^{x=2} \tfrac{1}{4}(x^3 - 6x^2 + 11x - 6)\Delta_2 x$$

$$-\underset{n_3=\infty}{\text{limit}} \sum_{x=2}^{x=3} \tfrac{1}{4}(x^3 - 6x^2 + 11x - 6)\Delta_3 x$$

$$+\underset{n_4=\infty}{\text{limit}} \sum_{x=3}^{x=4} \tfrac{1}{4}(x^3 - 6x^2 + 11x - 6)\Delta_4 x$$

$$= \tfrac{1}{4}\left(\tfrac{9}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{9}{4}\right) = \tfrac{5}{4}.$$

170. We have already seen that $\underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} f(x)\Delta x$ denotes an area or the negative of an area according as $f(x)$ is positive or negative for all values of x between a and b. This raises the question as to what it would denote if $f(x)$ changed in sign as x increases from a to b. In Art. 161, we saw that, in the particular case considered there, $\sum_{x=a}^{x=b} f(x)\Delta x$ denotes the sum of the areas of the rectangles from A to G, *minus* the sum of the areas of the rectangles from G to H, *plus* the sum of the areas of the rectangles from H to B. (Fig. 58.) It can be shown without much difficulty, although the proof will not be given here, that $\underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} f(x)\Delta x$ denotes the sum of the areas

of the regions ACK and MDB, *minus* the area of the region KFM. In general, $\underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} f(x)\Delta x$ denotes the sum of the areas of all the regions above the x-axis, *minus* the sum of the areas of all the regions below.

171. To evaluate $\underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} f(x)\Delta x$ when $f(x)$ changes in sign as x increases from a to b.

By Art. 147, the sum of the areas of all the regions above the x-axis, inclosed by the curve, the x-axis, and the ordinates corresponding to the abscissas a and b respectively, *minus* the sum of the areas of all the regions below, $= \phi(b) - \phi(a)$, where $\int f(x)\,dx = \phi(x) + c$. By the preceding article, this difference also $= \underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} f(x)\Delta x$. Therefore $\underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} f(x)\Delta x = \phi(b) - \phi(a)$, where $\int f(x)\,dx = \phi(x) + c$, if $f(x)$ is single valued and continuous for all values of x between and including a and b, and changes in sign as x increases from a to b.

172. From Arts. 166 and 171, we therefore see that

$$\underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} f(x)\Delta x = \phi(b) - \phi(a),$$

where $\int f(x)\,dx = \phi(x) + c$, if $f(x)$ is single valued and continuous for all values of x between and including a and b, whether it changes in sign as x increases from a to b or not.

173. Definitions. $\underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} f(x)\Delta x$ is written $\int_a^b f(x)\,dx$ and read, "the **definite integral** of $f(x)\,dx$ between a and b."

The values a and b are called the **lower** and **upper limits** of integration respectively.

The expression $f(x)$ is called the **integrand**.

174. In the last chapter, we saw that an indefinite integral could sometimes be evaluated by the substitution of a new variable. In a definite integral, if we substitute a new variable, we must either return to the old variable before substituting the limits in the result, or else change the limits at the time of substitution to suit the new variable.

For example, suppose that it is required to evaluate $\int_0^a \sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}}\, dx.$

Let $\qquad x^{\frac{1}{3}} = u. \qquad \therefore \frac{1}{3} x^{-\frac{2}{3}} dx = du.$

$$\therefore dx = 3u^2 du.$$

Substitute in the integral.

We may proceed as follows:

$$\int_0^a \sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}}\, dx$$

$$= 3\int_{x=0}^{x=a} u^2 \sqrt{a^{\frac{2}{3}} - u^2}\, du$$

$$= 3\left\{-\frac{u}{4}\sqrt{(a^{\frac{2}{3}} - u^2)^3} + \frac{a^{\frac{2}{3}}}{8}\left(u\sqrt{a^{\frac{2}{3}} - u^2} + a^{\frac{2}{3}} \sin^{-1}\frac{u}{a^{\frac{1}{3}}}\right)\right\}\Bigg|_{x=0}^{x=a}$$

$$= 3\left\{-\frac{x^{\frac{1}{3}}}{4}\sqrt{(a^{\frac{2}{3}} - x^{\frac{2}{3}})^3} + \frac{a^{\frac{2}{3}}}{8}\left(x^{\frac{1}{3}}\sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}} + a^{\frac{2}{3}} \sin^{-1}\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}}\right)\right\}\Bigg|_0^a$$

$$= \tfrac{3}{16}\pi a^{\frac{4}{3}}.$$

Or, we may proceed as follows:

Since $x^{\frac{1}{3}} = u$, $\therefore u = 0$ when $x = 0$, and $u = a^{\frac{1}{3}}$ when $x = a$. And, as x increases from 0 to a, u increases from 0 to $a^{\frac{1}{3}}$.

$$\therefore \int_0^a \sqrt{a^{\frac{2}{3}} - x^{\frac{2}{3}}}\, dx$$

$$= 3\int_0^{a^{\frac{1}{3}}} u^2 \sqrt{a^{\frac{2}{3}} - u^2}\, du$$

$$= 3\left\{-\frac{u}{4}\sqrt{(a^{\frac{2}{3}} - u^2)^3} + \frac{a^{\frac{2}{3}}}{8}\left(u\sqrt{a^{\frac{2}{3}} - u^2} + a^{\frac{2}{3}} \sin^{-1}\frac{u}{a^{\frac{1}{3}}}\right)\right\}\Bigg|_0^{a^{\frac{1}{3}}}$$

$$= \tfrac{3}{16}\pi a^{\frac{4}{3}}.$$

Of these two methods of procedure the latter is preferable, because the work of transforming back again to the old variable is usually more difficult than that of transforming the limits to suit the new variable.

EXERCISES

1. An ellipse whose semi-major and semi-minor axes are 5 feet and 3 feet respectively is cut into two parts by a line perpendicular to the major axis, 2 feet from the center. Find the area of each of the two segments of the ellipse.

Ans. 35.23 sq. ft.; 11.89 sq. ft.

2. A circle with its center at the origin, and a parabola with its vertex at the origin and axis on the x-axis both pass through the point $(2, 2\sqrt{2})$. Find the areas inclosed by the curves.

Ans. 24.35; 13.35.

3. A circle has a radius of 4 feet. A parabola has its vertex at the center of the circle, and the distance of its focus from the vertex is $1\frac{1}{2}$ feet. Find the smaller area inclosed by the two curves. *Ans.* 19.07 sq. ft.

In each of the five following curves, find the area inclosed by the curve, the x-axis, and the ordinates corresponding to the abscissas set opposite the equation:

4. $y = \sin^2 x.$ $x = 3,\ x = 4.$ *Ans.* 0.183.

5. $\dfrac{x^2}{36} + \dfrac{y^2}{4} = 1.$ $x = 3,\ x = 6.$ *Ans.* 7.37.

6. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1.$ $x = 0,\ x = 2.$ *Ans.* 11.48.

7. $y = x(x+1)(x+2).$ $x = -3,\ x = 3.$ *Ans.* 59.

8. $y = \frac{1}{12}(x-1)(x-3)(x-5).$ $x = -2,\ x = 7.$ *Ans.* 12.854.

9. $y = \log_{10} x.$ $x = \frac{1}{2},\ x = 2.$ *Ans.* 0.234.

10. Find the area of the segment of the circle $x^2 + y^2 = a^2$ cut off by the line $x = \dfrac{a}{2}$. *Ans.* $0.614\, a^2$.

11. Find the area inclosed between the parabola $x^2 = 4\,ay$ and the witch $y = \dfrac{8\,a^3}{x^2 + 4\,a^2}$. *Ans.* $(2\,\pi - \frac{4}{3})\,a^2$.

Evaluate the five following integrals, changing the limits as you change the variable:

12. $\displaystyle\int_3^5 \frac{(x-3)^{\frac{2}{3}}\,dx}{(x-3)^{\frac{2}{3}} + 4}$. Let $(x-3)^{\frac{1}{3}} = u$. *Ans.* 0.37.

13. $\displaystyle\int_0^2 \frac{\sqrt{e^x}\,dx}{e^x + 1}$. Let $e^x = u^2$. *Ans.* 2.44.

14. $\displaystyle\int_4^9 \frac{dx}{1 - \sqrt{x}}$. Let $\sqrt{x} = u$. *Ans.* -3.386..

15. $\displaystyle\int_1^{64} \frac{dx}{2\,x^{\frac{1}{2}} + x^{\frac{1}{3}}}$. Let $x^{\frac{1}{6}} = u$. *Ans.* 5.31.

16. $\displaystyle\int_{\frac{1}{2}}^{\frac{3}{2}} \sin 2\,x\,dx$. Let $2\,x = u$. *Ans.* 0.765.

17. Evaluate $\displaystyle\int_{\frac{1}{2}}^2 \cos(2\,x - 1)\,dx$. Find the area inclosed by the curve $y = \cos(2\,x - 1)$, the x-axis, and the ordinates corresponding to the abscissas $\frac{1}{2}$ and 2 respectively.

Ans. 0.071; 0.929.

Find the area of the loop in each of the three following curves:

18. $xy^2 = (x-a)^2(2\,a - x)$. *Ans.* $\frac{1}{2}(4 - \pi)\,a^2$.

19. $9\,y^2 = (x+7)(x+4)^2$. *Ans.* $\frac{8}{5}\sqrt{3}$.

20. $a^2y^2 = x^2(a-x)(2\,a-x)$. *Ans.* $\{\frac{11}{12}\sqrt{2} - \frac{3}{4}\log(\sqrt{2}+1)\}\,a^2$.

21. Find the larger area inclosed by the circle $x^2 + y^2 = 12\,x$ and the parabola $y^2 = 6\,x$. *Ans.* $18\,\pi + 48$.

22. Find the area cut off from the curve $27\,ay^2 = 4(x - 2\,a)^3$ by the line $x = 3\,a$. *Ans.* $\dfrac{8\,a^2}{15\sqrt{3}}$.

23. Find the area of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. *Ans.* $3\,\pi a^2$.

24. Find the area inclosed between the y-axis and the arc of the involute of the circle $x^2+y^2=a^2$ which is traced as ϕ varies from 0 to $\frac{\pi}{2}$. The equations of the involute are $x=a(\cos\phi+\phi\sin\phi)$, $y=a(\sin\phi-\phi\cos\phi)$. *Ans.* $\frac{\pi a^2}{4}\left(\frac{\pi^2}{12}+1\right)$.

175. We have supposed thus far in this chapter that $f(x)$ in the equation $y=f(x)$ is single valued and continuous for all values of x between and including a and b. We shall now suppose that $f(x)$ is single valued for all values of x between a and b, but violates the above condition of continuity by becoming infinite or infinite negatively either for $x=a$ or $x=b$ or some value or values of x between a and b, or by any combination of these.

Under these suppositions, we shall consider but one case, that in which $f(x)$ is single valued and continuous for all values of x between a and b, is continuous when $x=a$, and becomes infinite as x approaches b being always less than b. From the reasoning in this case, the student can readily supply the reasoning in any of the other cases.

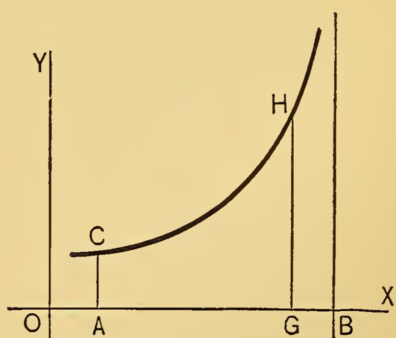


FIG. 64.

In the case just mentioned, the curve may be as in Fig. 64.

Let OA and OB represent the abscissas a and b respectively. On OB lay off a distance OG such that GB is equal to some arbitrary length ϵ. Then OG is of length $b-\epsilon$. Draw the ordinate GH.

$$\text{The area } AGHC = \lim_{n=\infty} \sum_{x=a}^{x=b-\epsilon} f(x)\,\Delta x$$

$$= \int_a^{b-\epsilon} f(x)\,dx.$$

Suppose that $$\int f(x)\,dx = \phi(x) + c.$$

Then $$\int_a^{b-\epsilon} f(x)\,dx = \phi(b-\epsilon) - \phi(a).$$

If $\phi(b-\epsilon) - \phi(a)$ approaches a limit as ϵ approaches zero, this limit is called the area between the curve, the x-axis, the ordinate AC, and the asymptote $x=b$. If $\phi(b-\epsilon)-\phi(a)$ increases without limit as ϵ approaches zero, the area between the curve, the x-axis, the ordinate AC, and the asymptote $x=b$, is said to be infinite.

176. Example 1. Find the area inclosed by the curve $y=\dfrac{1}{\sqrt[3]{x-1}}$, the x-axis, the ordinate corresponding to the abscissa -1, and the asymptote $x=1$.

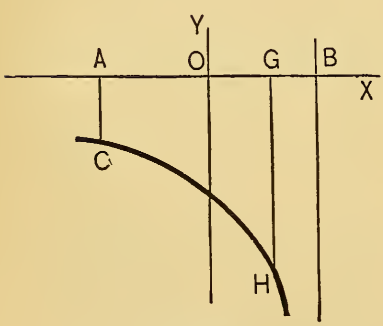


Fig. 65.

The curve is as in Fig. 65.

Let OA and OB represent the abscissas -1 and $+1$ respectively.

Let $GB=\epsilon$.

The area $AGHC$

$$= -\int_{-1}^{1-\epsilon}\frac{dx}{\sqrt[3]{x-1}}$$

$$= -\tfrac{3}{2}(x-1)^{\frac{2}{3}}\Big|_{-1}^{1-\epsilon}$$

$$= -\tfrac{3}{2}\epsilon^{\frac{2}{3}} + \tfrac{3}{2}2^{\frac{2}{3}}.$$

As ϵ approaches zero, $-\frac{3}{2}\epsilon^{\frac{2}{3}}+\frac{3}{2}2^{\frac{2}{3}}$ approaches $\frac{3}{2}2^{\frac{2}{3}}$.

Therefore the required area $=\frac{3}{2}2^{\frac{2}{3}}$.

Example 2. Find the area inclosed by the curve $y=\dfrac{1}{(x-1)^3}$, the x-axis, the ordinate corresponding to the abscissa -1, and the asymptote $x=1$.

Let OA and OB represent the abscissas -1 and $+1$ respectively (Fig. 66).

The area $AGHC$

$$= -\int_{-1}^{1-\epsilon} \frac{dx}{(x-1)^3}$$

$$= \frac{1}{2(x-1)^2}\Bigg|_{-1}^{1-\epsilon}$$

$$= \frac{1}{2\,\epsilon^2} - \frac{1}{8}.$$

As ϵ approaches zero, $\frac{1}{2\,\epsilon^2} - \frac{1}{8}$ becomes infinite.

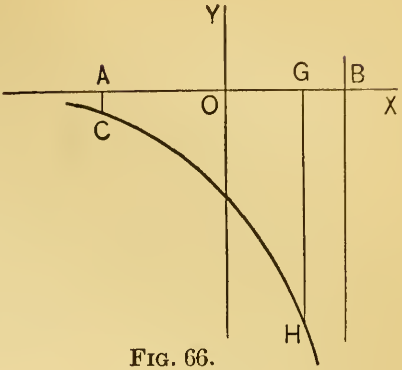


FIG. 66.

177. In Fig. 64, as ϵ approaches zero, $\sum\limits_{x=a}^{x=b-\epsilon} f(x)\,\Delta x$ approaches the sum of the rectangles inscribed on an equal number of equal divisions of the line AB. Then if $\underset{\Delta x \doteq 0}{\text{limit}} \sum\limits_{x=a}^{x=b-\epsilon} f(x)\,\Delta x$ approaches a limit as ϵ approaches zero, this limit can also be denoted by $\underset{\Delta x \doteq 0}{\text{limit}} \sum\limits_{x=a}^{x=b} f(x)\Delta x$, or $\int^b f(x)\,dx$.

EXERCISES

Evaluate the following integrals:

1. $\int_0^1 \frac{x^3\,dx}{\sqrt{1-x^2}}$. *Ans.* $\frac{2}{3}$.

2. $\int_0^1 \frac{x^4\,dx}{\sqrt{1-x^2}}$. *Ans.* $\frac{3\,\pi}{16}$.

3. $\int_0^1 \frac{x\,dx}{\sqrt{1-x^2}}$. *Ans.* 1.

4. $\int_0^1 \frac{dx}{1-x^2}$. *Ans.* Infinite.

5. $\int_0^\infty \frac{dx}{a^2+b^2x^2}$. *Ans.* $\frac{\pi}{2\,ab}$.

6. $\int_0^\infty xe^{-ax}\,dx$. *Ans.* $\frac{1}{a^2}$.

7. Find the area inclosed by the curve $y^3(x^2-a^2)^2=8x^3$, the x-axis, and the asymptote $x=a$. *Ans.* $3a^{\frac{2}{3}}$.

8. Find the area inclosed by the curve $y^3(x^2-a^2)^2=8x^3$, the x-axis, and the ordinate whose abscissa is $3a$. *Ans.* $9a^{\frac{2}{3}}$.

9. Find the area inclosed by the curve $y(x-a)^2=1$, the x-axis, and the ordinate whose abscissa is $2a$. *Ans.* Infinite.

10. Find the entire area inclosed by the witch $y=\dfrac{8a^3}{x^2+4a^2}$ and the x-axis.

Ans. $4\pi a^2$.

11. Find the area inclosed by the cissoid $y^2=\dfrac{x^3}{2a-x}$, the x-axis, and the asymptote $x=2a$.

Ans. $\dfrac{3}{2}\pi a^2$.

12. Find the area between the curve $xy=1$ and the x-axis.

Ans. Infinite.

CHAPTER XX

SUMMATION OF $\frac{1}{2}\{f(\theta)\}^2\Delta\theta$. PLANE AREAS IN POLAR COÖRDINATES

178. Let $r = f(\theta)$ be an equation in which $f(\theta)$ is single valued and continuous for all values of θ between and including two values α and β. Let $\dfrac{\beta - \alpha}{n} = \Delta\theta$, where n is any positive integer.

Definition. The symbol $\displaystyle\sum_{\theta=\alpha}^{\theta=\beta} \tfrac{1}{2}\{f(\theta)\}^2\Delta\theta$ is used to denote

$$\tfrac{1}{2}\{f(\alpha)\}^2\Delta\theta + \tfrac{1}{2}\{f(\alpha + \Delta\theta)\}^2\Delta\theta + \tfrac{1}{2}\{f(\alpha + 2\,\Delta\theta)\}^2\Delta\theta + \cdots$$
$$+ \tfrac{1}{2}\{f(\alpha + (n-1)\Delta\theta)\}^2\Delta\theta$$

for any value of n a positive integer.

To interpret $\displaystyle\sum_{\theta=\alpha}^{\theta=\beta} \tfrac{1}{2}\{f(\theta)\}^2\Delta\theta$ geometrically.

179. At first suppose that $f(\theta)$ is positive for all values of θ between α and β.

Under this supposition, the curve $r = f(\theta)$ may be as in Fig. 67.

Let OB and OC represent the radii vectores whose angles are α and β respectively. Divide the angle BOC into n equal parts. Each part has therefore the value $\dfrac{\beta - \alpha}{n}$.

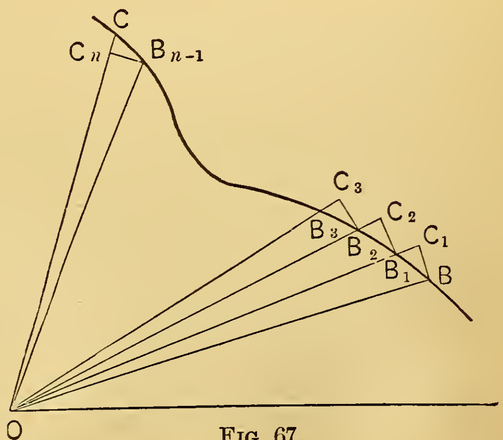


FIG. 67.

Call each part $\Delta\theta$. Denote the points at which the lines that divide the angle BOC into n equal parts meet the curve by B_1, $B_2, B_3, \cdots, B_{n-1}$. With center O and radii equal to OB, OB_1, $OB_2, \cdots, OB_{n-1}$, describe arcs of circles to meet the next succeeding radii vectores in $C_1, C_2, C_3, \cdots, C_n$ respectively.

The symbol $\sum\limits_{\theta=\alpha}^{\theta=\beta} \frac{1}{2}\{f(\theta)\}^2\Delta\theta$ denotes the sum of the areas of the circular sectors OBC_1, OB_1C_2, OB_2C_3, $\cdots$, $OB_{n-1}C_n$.

180. Next, suppose that $f(\theta)$ is negative for all values of θ between α and β, or changes in sign as θ increases from α to β.

Since $f(\theta)$ is real, $\frac{1}{2}\{f(\theta)\}^2$ is positive for all values of θ between α and β. Then, whether $f(\theta)$ is positive or negative, the area of each circular sector, described as explained in the preceding article, is $\frac{1}{2}\{f(\theta)\}^2\Delta\theta$. Therefore in these cases also, $\sum\limits_{\theta=\alpha}^{\theta=\beta} \frac{1}{2}\{f(\theta)\}^2\Delta\theta$ denotes the sum of the areas of circular sectors.

EXERCISES

In each of the following problems, plot the curve, draw the circular sectors, and calculate:

1. $\sum\limits_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{2}(1-\cos\theta)^2\Delta\theta$, when $n=4$. *Ans.* 0.093.

2. $\sum\limits_{\theta=0}^{\theta=\frac{\pi}{3}} \sin\theta\,\Delta\theta$, when $n=5$. *Ans.* 0.408.

181. Let $r=f(\theta)$ be an equation in which $f(\theta)$ is single valued and continuous for all values of θ between and including two values α and β. To find the area inclosed by the curve $r=f(\theta)$, and the radii vectores that make angles of α and β respectively with the initial line.

Suppose that the curve is as in Fig. 68.

Let OB and OC represent the radii vectores whose angles are α and β respectively. Divide the angle BOC into n equal

parts. Call each part $\Delta\theta$. Denote the points at which the lines that divide the angle BOC into n equal parts meet the curve by $B_1, B_2, B_3, \cdots, B_{n-1}$. Let the radii vectores OB_1, OB_2, OB_3, $\cdots$, OB_{n-1}, make the angles θ_1, θ_2, θ_3, $\cdots$, θ_n respectively, with the initial line.

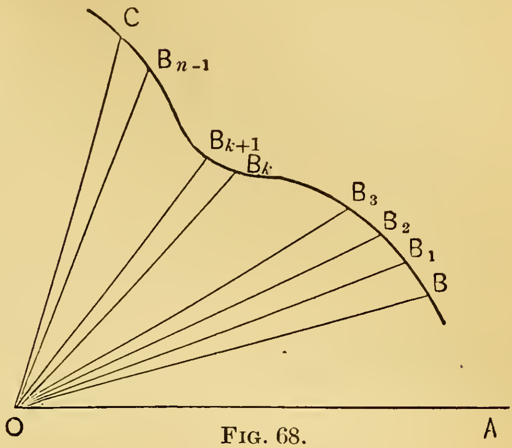


FIG. 68.

The radii vectores divide the area up into infinitesimal strips of area. Let $\Delta_k A$ represent a strip $OB_k B_{k+1}$. Then, by Art. 149,

$$\underset{\Delta\theta \doteq 0}{\text{limit}} \left[\frac{\Delta_k A}{\Delta\theta}\right] = \tfrac{1}{2}\{f(\theta_k)\}^2.$$

Therefore, from the definition of a limit,

$\dfrac{\Delta_k A}{\Delta\theta} = \tfrac{1}{2}\{f(\theta_k)\}^2 + \epsilon_k$ where ϵ_k is infinitesimal as $\Delta\theta \doteq 0$, or when $n = \infty$.

$$\therefore\ \Delta_k A = \tfrac{1}{2}\{f(\theta_k)\}^2\Delta\theta + \epsilon_k\Delta\theta.$$

Let k take in succession the values $1, 2, 3, \cdots, n-1$. Let $\Delta_k A$ and ϵ_k when $\theta = \alpha$ be denoted by $\Delta_0 A$ and ϵ_0 respectively.

Then
$$\Delta_0 A = \tfrac{1}{2}\{f(\alpha)\}^2\Delta\theta + \epsilon_0\Delta\theta,$$
$$\Delta_1 A = \tfrac{1}{2}\{f(\theta_1)\}^2\Delta\theta + \epsilon_1\Delta\theta,$$
$$\Delta_2 A = \tfrac{1}{2}\{f(\theta_2)\}^2\Delta\theta + \epsilon_2\Delta\theta,$$
$$\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots$$
$$\Delta_{n-1} A = \tfrac{1}{2}\{f(\theta_{n-1})\}^2\Delta\theta + \epsilon_{n-1}\Delta\theta.$$

Then, as in Art. 164, $A = \underset{n=\infty}{\text{limit}} \displaystyle\sum_{\theta=\alpha}^{\theta=\beta} \tfrac{1}{2}\{f(\theta)\}^2\Delta\theta.$

Therefore the required area $= \underset{n=\infty}{\text{limit}} \displaystyle\sum_{\theta=\alpha}^{\theta=\beta} \tfrac{1}{2}\{f(\theta)\}^2\Delta\theta.$

182. To evaluate $\underset{n=\infty}{\text{limit}} \sum_{\theta=\alpha}^{\theta=\beta} \frac{1}{2}\{f(\theta)\}^2\Delta\theta$:

It was shown in Art. 149 that the required area

$$= \phi(\beta) - \phi(\alpha), \text{ where } \tfrac{1}{2}\int\{f(\theta)\}^2 d\theta = \phi(\theta) + c.$$

Therefore $$\underset{n=\infty}{\text{limit}} \sum_{\theta=\alpha}^{\theta=\beta} \tfrac{1}{2}\{f(\theta)\}^2\Delta\theta = \phi(\beta) - \phi(\alpha),$$

where $$\tfrac{1}{2}\int\{f(\theta)\}^2 d\theta = \phi(\theta) + c.$$

183. Definitions. $\underset{n=\infty}{\text{limit}} \sum_{\theta=\alpha}^{\theta=\beta} \frac{1}{2}\{f(\theta)\}^2\Delta\theta$ is denoted by

$$\tfrac{1}{2}\int_{\alpha}^{\beta}\{f(\theta)\}^2 d\theta,$$

and read, "one half the definite integral of $\{f(\theta)\}^2 d\theta$ between α and β."

The values α and β are called the lower and upper limits of integration respectively. The expression $\{f(\theta)\}^2$ is called the integrand.

184. Example. Find the area inclosed by the curve $r = a(1 - \cos\theta)$, and the radii vectores that make angles of $0°$ and $180°$ respectively with the initial line.

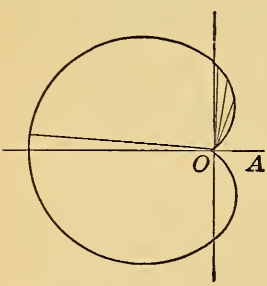


Fig. 69.

The curve is as in Fig. 69.

The required area

$$= \underset{n=\infty}{\text{limit}} \sum_{\theta=0}^{\theta=\pi} \tfrac{1}{2}a^2(1 - \cos\theta)^2\Delta\theta$$

$$= \frac{a^2}{2}\int_0^{\pi}(1 - \cos\theta)^2 d\theta$$

$$= \tfrac{3}{4}\pi a^2.$$

EXERCISES

In each of the three following curves, find the area inclosed by the curve, and the radii vectores that make with the initial line the angles θ set opposite the equation.

1. $r = 4\cos\theta$. $\theta = 0$, $\theta = \frac{1}{2}$. *Ans.* 3.683.

2. $r = 6\sin\theta$. $\theta = 0$, $\theta = \dfrac{\pi}{4}$. *Ans.* 2.569.

3. $r = 4\cos 2\theta$. $\theta = 0$, $\theta = \frac{1}{2}$. *Ans.* 2.909.

4. Find the area inclosed by one loop of the curve $r = a\sin 2\theta$.

Ans. $\dfrac{\pi a^2}{8}$.

5. Find the area inclosed by the curve $r = a\sin^3\dfrac{\theta}{3}$ and the initial line, below the initial line.

Ans. $(10\pi + 27\sqrt{3})\dfrac{a^2}{64}$.

6. Find the area inclosed by a small loop of the curve $r = a\sin\frac{1}{2}\theta$. Also the area inclosed by the whole curve.

Ans. $\frac{1}{4}(\pi - 2)a^2$; $\frac{1}{2}(\pi + 2)a^2$.

7. Find the area inclosed by the loop of the curve $r^2\cos\theta = a^2\sin 3\theta$ *Ans.* $\dfrac{3a^2}{4} - \dfrac{a^2}{2}\log 2$.

8. Find the area inclosed by a small loop of the curve $r^2 = a^2\cos\dfrac{\theta}{2}$. *Ans.* $(2 - \sqrt{2})a^2$.

9. Find the area of a loop of the curve $r^2 = a^2\cos 2\theta$.

Ans. $\dfrac{a^2}{2}$.

10. Find the area inclosed by the curves $r = \dfrac{4}{1 - \cos\theta}$ and $r = \dfrac{4}{1 + \cos\theta}$. *Ans.* $\frac{64}{3}$.

11. Find the area inclosed by the curve $r = a(\sin 2\theta + \cos 2\theta)$.

Ans. πa^2.

12. Find the area inclosed by the outer branch of the curve $r = 1 + 2\sin\frac{3}{2}\theta$ and the initial line, above the initial line.

Ans. $\dfrac{7}{6}\pi + \dfrac{\sqrt{3}}{2} + \dfrac{4}{3}$.

13. Find the area of a loop of the curve $r = 2a\sin 3\theta$.

Ans. $\frac{1}{3}\pi a^2$.

CHAPTER XXI

THEOREM IN INFINITESIMALS. DEFINITE INTEGRAL IN GENERAL

185. As already seen, each term of $\sum\limits_{x=a}^{x=b} f(x)\Delta x$ of Chapter XIX is the area or the negative of the area of a rectangle, two of whose sides remain finite while the other two approach zero as a limit as n increases without limit. Also, each term of $\sum\limits_{\theta=\alpha}^{\theta=\beta} \frac{1}{2}\{f(\theta)\}^2\Delta\theta$ of Chapter XX is the area of a circular sector, the radii of which remain finite while the angle between them approaches zero as a limit as n increases without limit. The area, therefore, in either case is expressed as the limit of a sum of terms, each of which is infinitesimal as the number of terms increases without limit. The student will see later that many of the problems discussed in the remaining chapters of the book also involve the limit of a sum of terms, each of which is infinitesimal as the number of terms increases without limit.

186. The limit of a sum of terms, each of which is infinitesimal as the number of terms increases without limit, can frequently be written more simply by the aid of the following theorem in infinitesimals.

Theorem. Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$, be n infinitesimals, all of the same sign, such that $\underset{n=\infty}{\text{limit}}\left[\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n\right]=A$, where A is some definite number. Let $\beta_1, \beta_2, \beta_3, \cdots, \beta_n$, be n other infinitesimals such that $\underset{n=\infty}{\text{limit}}\left[\frac{\beta_1}{\alpha_1}\right]=1$, $\underset{n=\infty}{\text{limit}}\left[\frac{\beta_2}{\alpha_2}\right]=1$, $\underset{n=\infty}{\text{limit}}\left[\frac{\beta_3}{\alpha_3}\right]=1$, $\cdots$, $\underset{n=\infty}{\text{limit}}\left[\frac{\beta_n}{\alpha_n}\right]=1$. Then must $\underset{n=\infty}{\text{limit}}\left[\beta_1+\beta_2+\beta_3+\cdots+\beta_n\right]=A$.

Proof. Of the ratios $\frac{\beta_1}{\alpha_1}, \frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3}, \cdots, \frac{\beta_n}{\alpha_n}$, let $\frac{\beta_k}{\alpha_k}$ be one which is not greater, and $\frac{\beta_r}{\alpha_r}$ one which is not less than any of the others. Then, by a familiar theorem of algebra,

$$\frac{\beta_k}{\alpha_k} \leqq \frac{\beta_1+\beta_2+\beta_3+\cdots+\beta_n}{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n} \leqq \frac{\beta_r}{\alpha_r}.$$

As n increases without limit, $\frac{\beta_k}{\alpha_k}$ and $\frac{\beta_r}{\alpha_r}$ each approaches the limit 1. Therefore, since $\frac{\beta_1+\beta_2+\beta_3+\cdots+\beta_n}{\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n}$ is always greater than the one and less than the other, it also approaches the limit 1. And, by supposition,

$$\operatorname*{limit}_{n=\infty}\left[\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n\right]=A.$$

Therefore $$\operatorname*{limit}_{n=\infty}\left[\beta_1+\beta_2+\beta_3+\cdots+\beta_n\right]=A.$$

187. As an application of this theorem, we shall again prove that the area defined in Art. 163 is $\operatorname*{limit}_{n=\infty}\sum_{x=a}^{x=b} f(x)\Delta x$.

Make the same construction as in Fig. 59. See Fig. 70. Denote the abscissas OA_1, OA_2, OA_3, $\cdots$, OA_{n-1}, by x_1, x_2, x_3, $\cdots$, x_{n-1} respectively.

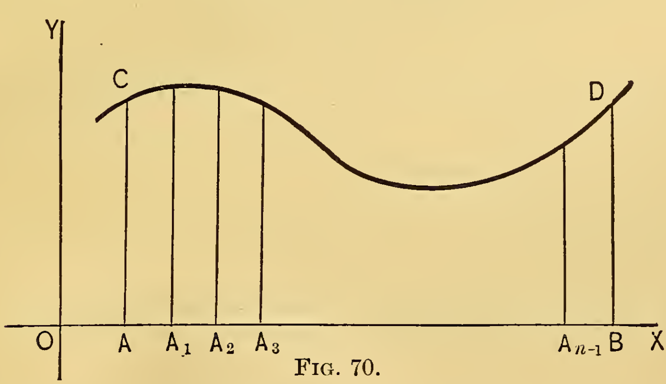


FIG. 70.

The required area is the sum of the strips whose bases are AA_1, A_1A_2, A_2A_3, $\cdots$, $A_{n-1}B$. Denote these strips by α_1, α_2, α_3, $\cdots$, α_n.

Therefore the required area $= \alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n$

$$= \underset{n=\infty}{\text{limit}} \left[\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n\right], \quad (1)$$

where each strip is infinitesimal as the number of strips increases without limit.

Let α_{k+1} denote any one of these strips. Let x'_k and x''_k be the abscissas corresponding to the longest and shortest ordinates that can be drawn in this strip.

Then $$f(x''_k)\Delta x < \alpha_{k+1} < f(x'_k)\Delta x.$$

Divide by $f(x_k)\Delta x$.

$$\therefore \frac{f(x''_k)\Delta x}{f(x_k)\Delta x} < \frac{\alpha_{k+1}}{f(x_k)\Delta x} < \frac{f(x'_k)\Delta x}{f(x_k)\Delta x}.$$

$$\therefore \frac{f(x''_k)}{f(x_k)} < \frac{\alpha_{k+1}}{f(x_k)\Delta x} < \frac{f(x'_k)}{f(x_k)}.$$

As $\Delta x \doteq 0$, or when $n = \infty$, $f(x'_k)$ and $f(x''_k)$ both approach $f(x_k)$.

$$\therefore \underset{n=\infty}{\text{limit}} \left[\frac{f(x''_k)}{f(x_k)}\right] = 1, \text{ and } \underset{n=\infty}{\text{limit}} \left[\frac{f(x'_k)}{f(x_k)}\right] = 1.$$

$$\therefore \underset{n=\infty}{\text{limit}} \left[\frac{\alpha_{k+1}}{f(x_k)\Delta x}\right] = 1.$$

Therefore, by the theorem of the preceding article, α_{k+1} may be replaced by $f(x_k)\Delta x$ in any problem involving the limit of the sum of the infinitesimals α. Let α_{k+1} take in succession the values $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$, and x_k the values $a, x_1, x_2, \cdots, x_{n-1}$. Substitute in (1).

Therefore the required area $= \underset{n=\infty}{\text{limit}} \Big[f(a)\Delta x + f(x_1)\Delta x$

$$+ f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x\Big].$$

$$= \underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} f(x)\Delta x.$$

DEFINITE INTEGRAL IN GENERAL

188. In $\sum_{x=a}^{x=b} f(x)\Delta x$ of Chapter XIX, each term is built up from $f(x)$ by multiplying the proper value of $f(x)$ for that term by Δx. In $\sum_{\theta=\alpha}^{\theta=\beta}\frac{1}{2}\{f(\theta)\}^2\Delta\theta$ of Chapter XX, each term is built up from $f(\theta)$ by squaring the proper value of $f(\theta)$ for that term and multiplying the result by $\frac{1}{2}\Delta\theta$. The student will see later that in all problems involving the limit of the sum of a set of infinitesimal terms, each term is built up from the original function in some way. Let $F(a)\Delta x + F(a+\Delta x)\Delta x + F(a+2\,\Delta x)\Delta x + \cdots + F(a+(n-1)\Delta x)\Delta x$, where $\Delta x = \dfrac{b-a}{n}$, be a sum of terms built up from the original function in any way. Suppose that $F(x)$ is single valued and continuous for all values of x between and including a and b.

189. **Definitions.** The symbol $\sum_{x=a}^{x=b} F(x)\Delta x$ is used to denote
$F(a)\Delta x+F(a+\Delta x)\Delta x+F(a+2\,\Delta x)\Delta x+\cdots+F(a+(n-1)\Delta x)\Delta x$
for any value of n a positive integer.

$\lim_{n=\infty}\sum_{x=a}^{x=b} F(x)\Delta x$ is denoted by $\int_a^b F(x)\,dx$, and read, "the definite integral of $F(x)\,dx$ between a and b."

As in Art. 173, a and b are called the lower and upper limits of integration respectively.

The expression $F(x)$ is called the integrand.

190. To evaluate $\int_a^b F(x)\,dx$, when $F(x)$ is single valued and continuous for all values of x between and including a and b:

Plot the curve whose equation is $y=F(x)$.

The function $F(x)$ is some function of x, single valued and continuous for all values of x between and including a and b. The function $f(x)$ of Chapter XIX is any function of x which

satisfies the same conditions. The reasoning, therefore, applied in Chapter XIX to show that $\lim_{n=\infty} \sum_{x=a}^{x=b} f(x)\Delta x$, or $\int_a^b f(x)\,dx$, $= \phi(b) - \phi(a)$, where $\int f(x)dx = \phi(x) + c$, will apply equally well here to show that $\int_a^b F(x)dx = \phi(b) - \phi(a)$, where $\int F(x)dx = \phi(x) + c$.

Therefore, $\int_a^b F(x)dx = \phi(b) - \phi(a)$, where $\int F(x)dx = \phi(x) + c$.

EXERCISE

Prove by means of the theorem of Art. 186 that the area defined in Art. 181 is $\lim_{n=\infty} \sum_{\theta=\alpha}^{\theta=\beta} \frac{1}{2}\{f(\theta)\}^2\Delta\theta$.

CHAPTER XXII

SUMMATION OF $\sqrt{\Delta x^2 + \Delta y^2}$. LENGTH OF AN ARC OF A PLANE CURVE

191. Let $y = f(x)$ be an equation in which $f(x)$ is single valued and continuous for all values of x between and including two values a and b.

Suppose that the curve $y = f(x)$ is as in Fig. 71.

Let OA and OB represent the abscissas a and b respectively.

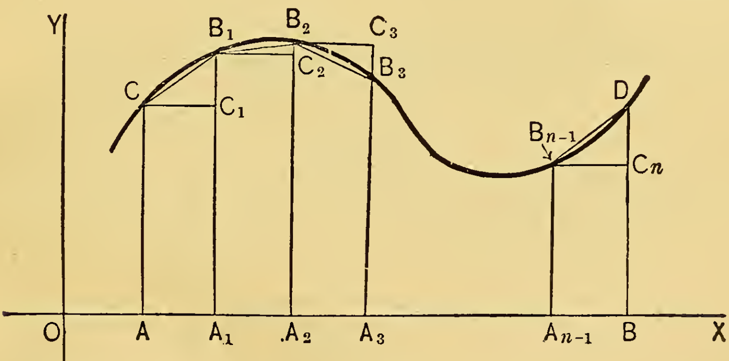


FIG. 71.

Divide AB into n equal parts. Call each part Δx. Denote the successive points of division of AB by $A_1, A_2, A_3, \cdots, A_{n-1}$. At $A, A_1, A_2, A_3, \cdots, A_{n-1}, B$, erect ordinates to meet the curve in $C, B_1, B_2, B_3, \cdots, B_{n-1}, D$ respectively. From $C, B_1, B_2, \cdots, B_{n-1}$, draw lines parallel to the x-axis to meet the next succeeding ordinates or ordinates produced in $C_1, C_2, C_3, \cdots, C_n$ respectively. Join $CB_1, B_1B_2, B_2B_3, \cdots, B_{n-1}D$.

Denote the abscissas of the points $A_1, A_2, A_3, \cdots, A_{n-1}$, by $x_1, x_2, x, \cdots, x_{n-1}$ respectively. Denote the chords CB_1, B_1B_2, $B_2B_3, \cdots, B_{n-1}D$ by $\Delta_1 c, \Delta_2 c, \Delta_3 c, \cdots, \Delta_n c$ respectively. Then

$$\Delta_1 c = \sqrt{\Delta x^2 + \Delta_1 y^2}, \text{ where } \Delta_1 y = f(x_1) - f(a),$$

$$\Delta_2 c = \sqrt{\Delta x^2 + \Delta_2 y^2}, \text{ where } \Delta_2 y = f(x_2) - f(x_1),$$

$$\Delta_3 c = \sqrt{\Delta x^2 + \Delta_3 y^2}, \text{ where } \Delta_3 y = f(x_3) - f(x_2),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta_n c = \sqrt{\Delta x^2 + \Delta_n y^2}, \text{ where } \Delta_n y = f(b) - f(x_{n-1}).$$

192. Definitions. The symbol $\sum\limits_{x=a}^{x=b} \sqrt{\Delta x^2 + \Delta y^2}$ is used to denote

$$\sqrt{\Delta x^2 + \Delta_1 y^2} + \sqrt{\Delta x^2 + \Delta_2 y^2} + \sqrt{\Delta x^2 + \Delta_3 y^2} + \cdots + \sqrt{\Delta x^2 + \Delta_n y^2}.$$

The symbol $\sum\limits_{x=a}^{x=b} \Delta c$ is used to denote $\Delta_1 c + \Delta_2 c + \Delta_3 c + \cdots + \Delta_n c$.

If $\Delta_1 s, \Delta_2 s, \Delta_3 s, \cdots, \Delta_n s$, denote the lengths of the arcs corresponding to $\Delta_1 c, \Delta_2 c, \Delta_3 c, \cdots, \Delta_n c$ respectively, the symbol $\sum\limits_{x=a}^{x=b} \Delta s$ is used to denote $\Delta_1 s + \Delta_2 s + \Delta_3 s + \cdots + \Delta_n s$.

EXERCISES

1. In the equation $y = x^3$, calculate $\sum\limits_{x=2}^{x=3} \sqrt{\Delta x^2 + \Delta y^2}$ when $n = 4$. *Ans.* 19.03.

2. In the equation $y = \sin x$, calculate $\sum\limits_{x=0}^{x=\frac{\pi}{2}} \sqrt{\Delta x^2 + \Delta y^2}$ when $n = 3$. *Ans.* 1.90.

RECTANGULAR COÖRDINATES

193. Let $y = f(x)$ be an equation in which $f(x)$ is single valued and continuous for all values of x between and including two values a and b. To find the length of the arc of the curve $y = f(x)$ contained between the two points on the curve whose abscissas are a and b respectively.

194. Before proceeding to a discussion of the problem stated in the preceding article, we shall first establish the following theorem :

Theorem. If an arc of a plane curve is concave toward its chord, the limit of the ratio of the arc to the chord as both approach zero is unity.

Let APB be an arc concave toward its chord AB (Fig. 72 or Fig. 73). At A and B draw tangents to the arc to meet in Q. From Q draw QM perpendicular to AB.

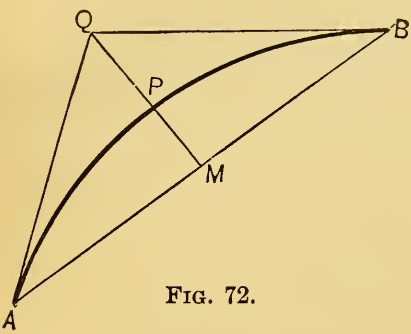


FIG. 72.

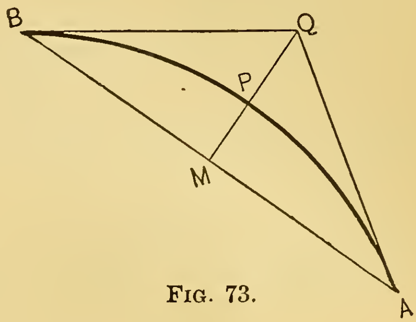


FIG. 73.

$$\frac{AQ}{AM} = \sec MAQ.$$

As AB approaches zero, $\angle MAQ$ approaches zero, and therefore $\sec MAQ$ approaches 1. $\therefore \underset{AB \doteq 0}{\text{limit}} \left[\frac{AQ}{AM}\right] = 1.$ Also, $\underset{AB \doteq 0}{\text{limit}} \left[\frac{QB}{MB}\right] = 1.$ Therefore, from the fundamental idea of a limit, $\frac{AQ}{AM} = 1 + \epsilon_1$ and $\frac{QB}{MB} = 1 + \epsilon_2$, where ϵ_1 and ϵ_2 are infinitesimal.

$$\therefore AQ = AM + AM \cdot \epsilon_1, \text{ and } QB = MB + MB \cdot \epsilon_2.$$

$$\therefore AQ + QB = AM + MB + AM \cdot \epsilon_1 + MB \cdot \epsilon_2$$

$$= AB + AM \cdot \epsilon_1 + MB \cdot \epsilon_2.$$

$$\therefore \frac{AQ + QB}{AB} = 1 + \frac{AM}{AB}\epsilon_1 + \frac{MB}{AB}\epsilon_2.$$

Since $\frac{AM}{AB} < 1$, and ϵ_1 approaches zero as AB approaches zero,

$$\lim_{AB \doteq 0}\left[\frac{AM}{AB}\cdot\epsilon_1\right]=0. \quad \text{Also,} \lim_{AB \doteq 0}\left[\frac{MB}{AB}\cdot\epsilon_2\right]=0.$$

$$\therefore \lim_{AB \doteq 0}\left[\frac{AQ+QB}{AB}\right]=1.$$

Now $$AB < \text{arc } APB < AQ + QB.$$

Divide by AB.

$$\therefore 1 < \frac{\text{arc } APB}{AB} < \frac{AQ+QB}{AB}.$$

Therefore $\frac{\text{arc } APB}{AB}$ is always greater than 1 and less than a number that approaches 1.

Therefore $$\lim_{AB \doteq 0}\left[\frac{\text{arc } APB}{AB}\right]=1.$$

195. To return to the problem of Art. 193.

Suppose that the curve $y=f(x)$ is concave toward all chords that can be drawn to it from points between the two whose abscissas are a and b respectively, and that the slope of the tangent line to the curve does not become infinite at any point between and including these two points.

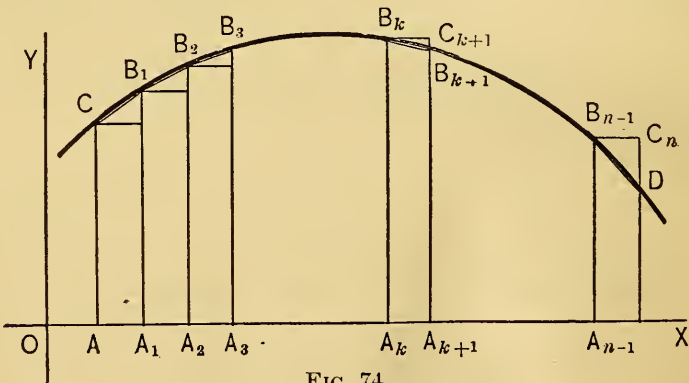


FIG. 74.

Under this supposition the curve may be as in Fig. 74. Make the same construction as in Fig. 71, Art. 191.

$$\text{The required length of arc} = \sum_{x=a}^{x=b} \Delta s$$

$$= \underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} \Delta s$$

$$= \underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} \Delta c,$$

by the preceding article, since $\underset{n=\infty}{\text{limit}} \left[\frac{\Delta s}{\Delta c}\right] = 1,$

$$= \underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} \sqrt{\Delta x^2 + \Delta y^2}. \tag{1}$$

The chord $B_k B_{k+1} = \sqrt{\Delta x^2 + \Delta_{k+1} y^2}$. Compare this length with $\sqrt{dx^2 + d_{k+1} y^2}$, where $d_{k+1} y$ is the differential corresponding to $\Delta_{k+1} y$.

$$\underset{n=\infty}{\text{limit}} \left[\frac{\sqrt{\Delta x^2 + \Delta_{k+1} y^2}}{\sqrt{dx^2 + d_{k+1} y^2}}\right] = \underset{n=\infty}{\text{limit}} \left[\frac{\sqrt{1 + \left(\frac{\Delta_{k+1} y}{\Delta x}\right)^2}}{\sqrt{1 + \left(\frac{d_{k+1} y}{dx}\right)^2}}\right]$$

$$= \frac{\sqrt{1 + \left(\frac{d_{k+1} y}{dx}\right)^2}}{\sqrt{1 + \left(\frac{d_{k+1} y}{dx}\right)^2}} = 1.$$

Therefore $\sqrt{\Delta x^2 + \Delta_{k+1} y^2}$ may be replaced by $\sqrt{dx^2 + d_{k+1} y^2}$ in any problem involving the limit of the sum of infinitesimals. Replace each term in (1) by $\sqrt{dx^2 + d_{k+1} y^2}$, where $d_{k+1} y$ has the proper value for that term.

Therefore the required length of arc

$$= \underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} \sqrt{dx^2 + dy^2} \tag{2}$$

$$= \underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

by dividing and multiplying by dx,

$$=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

The length of an arc can also be expressed as $\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$ by dividing and multiplying (2) by dy. In this case c and d are the values of y corresponding respectively to the values a and b of x.

196. The results of the preceding article have been established only in the case where the curve satisfies the conditions imposed in that article. When either or both of these conditions are violated, a special investigation is necessary. It would be found, however, that in any case in which the equation of the curve satisfies the conditions of Art. 193, the results are the same as those obtained in the case already considered. As the investigation would be tedious, we shall assume the truth of this statement without further discussion.

197. Example. Find the length of the arc of the catenary $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ contained between the two points on the curve whose abscissas are 0 and a respectively.

The curve is as in Fig. 75.

Since $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$, $\frac{dy}{dx}=\frac{1}{2}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)$.

Therefore the required length of arc

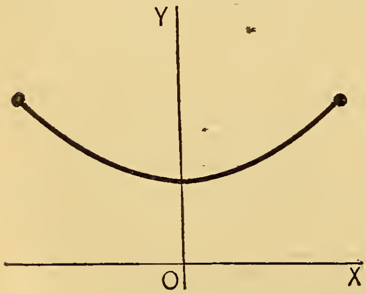


Fig. 75.

$$=\underset{n=\infty}{\text{limit}}\sum_{x=0}^{x=a}\sqrt{1+\tfrac{1}{4}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)^2}\,\Delta x$$

$$=\int_0^a \sqrt{1+\tfrac{1}{4}\left(e^{\frac{x}{a}}-e^{-\frac{x}{a}}\right)^2}\,dx$$

$$=\frac{1}{2}\int_0^a \left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)dx$$

$$=\frac{a}{2}\left(e-\frac{1}{e}\right).$$

POLAR COÖRDINATES

198. Let $r = f(\theta)$ be an equation in which $f(\theta)$ is single valued and continuous for all values of θ between and including two values α and β. To find the length of the arc of the curve $r = f(\theta)$ contained between the two points on the curve whose radii vectores make angles of α and β respectively with the initial line.

Suppose that the curve that represents the equation $r = f(\theta)$ is as in Fig. 76.

Let AOB and AOC represent the angles α and β respectively. From C and B draw CD and BG perpendicular to OA.

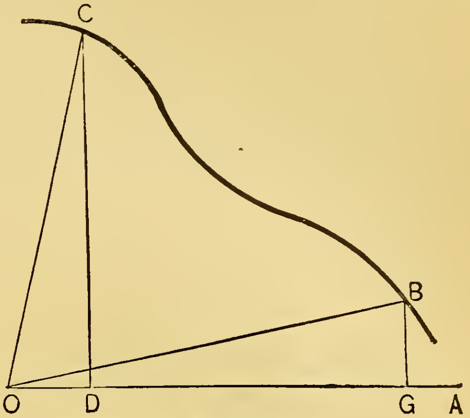


FIG. 76.

Suppose that B and C have the abscissas a and b respectively when the equation of the curve is expressed in rectangular coördinates. Then, in rectangular coördinates, the required length of arc is $\underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} \sqrt{dx^2 + dy^2}$. Transform this expression to polar coördinates. The equations of transformation are

$$x = r \cos \theta,$$

$$y = r \sin \theta.$$

Since $x = r \cos \theta$, $\therefore\ dx = \cos \theta\, dr - r \sin \theta\, d\theta$.

Since $y = r \sin \theta$, $\therefore\ dy = \sin \theta\, dr + r \cos \theta\, d\theta$.

$$\therefore \underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} \sqrt{dx^2 + dy^2} = \underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} \sqrt{dr^2 + r^2 d\theta^2}.$$

Now $\theta = \alpha$ when $x = a$, and $\theta = \beta$ when $x = b$.

$$\underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} \sqrt{dx^2 + dy^2} = \underset{n=\infty}{\text{limit}} \sum_{\theta=\alpha}^{\theta=\beta} \sqrt{dr^2 + r^2 d\theta^2}.$$

Therefore the required length of arc $= \underset{n=\infty}{\text{limit}} \sum_{\theta=\alpha}^{\theta=\beta} \sqrt{dr^2 + r^2 d\theta^2}$

$$= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

The length of the arc can also be expressed as

$$\int_\gamma^\delta \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$$

by dividing and multiplying by dr. In this case, γ and δ are the values of r corresponding to the values α and β respectively of θ.

199. Example. Find the entire length of the cardioide $r = a(1 - \cos\theta)$.

The curve is as in Fig. 77.

The required length of arc

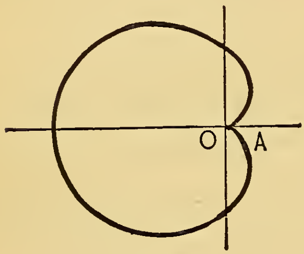


Fig. 77.

$$= 2\int_0^\pi \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}\, d\theta$$

$$= 2a\int_0^\pi \sqrt{2(1-\cos\theta)}\, d\theta$$

$$= 4a\int_0^\pi \sin\frac{\theta}{2}\, d\theta$$

$$= 8a.$$

EXERCISES

1. Find the length of the arc of the curve $y = \log x$ contained between the points on the curve whose abscissas are 1 and 5 respectively. *Ans.* 4.37.

2. Find the length of the four cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ contained between the points on the curve whose abscissas are $-\frac{1}{2}$ and 1 respectively. *Ans.* 2.445.

3. Find the length of the arc of the four cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, first using the limits 0 to a, then the limits $-a$ to a. *Ans.* $6a$.

4. Find the length of the loop of the curve

$$9y^2 = (x+7)(x+4)^2. \qquad \textit{Ans. } 4\sqrt{3}.$$

5. Find the length of the arc of the curve $y = \log \sec x$ contained between the points on the curve whose abscissas are 0 and $\frac{\pi}{3}$ respectively. *Ans.* $\log(2+\sqrt{3}) = 1.32.$

6. Find the length of the parabola $y^2 = 4ax$ cut off by the latus rectum. *Ans.* $2[\sqrt{2} + \log(1+\sqrt{2})]a = 4.591\,a.$

7. Find the length of the arc of the curve $y = \log(1-x^2)$ contained between the points on the curve whose abscissas are 0 and x respectively. *Ans.* $\log\frac{1+x}{1-x} - x.$

8. Find the length of the arc of the curve $y^3 = ax^2$ contained between the point (0, 0) and the point (x, y) on the curve.

Ans. $\frac{(4a+9y)^{\frac{3}{2}}}{27a^{\frac{1}{2}}}.$

9. Find the length of an arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$. *Ans.* $8a.$

10. Find the length of the arc traced by the extremity of a string while it is being unwound from a circle. Given that the equations of the involute of the circle are $x = a(\cos\phi + \phi\sin\phi)$, $y = a(\sin\phi - \phi\cos\phi)$. *Ans.* $2\pi^2 a.$

11. Find the length of an arc of the hypocycloid

$$x = (a-b)\cos\phi + b\cos\frac{a-b}{b}\phi,\ y = (a-b)\sin\phi - b\sin\frac{a-b}{b}\phi$$

described in a complete revolution of the generating circle.

Ans. $\frac{8b(a-b)}{a}.$

12. The cables of a suspension bridge hang in the form of a parabola. Given that the length of the bridge is 1000 feet and that the distance from the lowest point of the cable to the level of the top of the piers is 50 feet, find the length of a cable.

Ans. 1006.4 ft.

13. Find the length of the arc of the circle $r=2\,a\cos\theta$.

Ans. $2\,\pi a$.

14. Find the length of the arc of the hyperbolic spiral $r\theta=a$ contained between the points on the curve where $\theta=\frac{5}{12}$ and $\theta=\frac{3}{4}$ respectively.

Ans. $[\frac{14}{15}+\log\frac{4}{3}]\,a$.

15. Find the length of the arc of the spiral of Archimedes $r=a\theta$ contained between the points on the curve where $\theta=0$ and $\theta=2\,\pi$ respectively.

Ans. $[\pi\sqrt{1+4\,\pi^2}+\frac{1}{2}\log(2\,\pi+\sqrt{1+4\,\pi^2})]\,a$.

16. Find the length of the arc of the parabola $r=\dfrac{4}{1+\cos\theta}$, contained between the points on the curve where $\theta=0$ and $\theta=\dfrac{\pi}{2}$ respectively.

Ans. $2\left[\sqrt{2}+\log\tan\dfrac{3\,\pi}{8}\right]$.

17. Find the length of the arc of the curve $r^{\frac{1}{3}}=a^{\frac{1}{3}}\sin\dfrac{\theta}{3}$.

Ans. $\frac{3}{2}\,\pi a$.

18. Find the length of the arc of the cissoid $r=20\tan\theta\sin\theta$ contained between the cusp and the point on the curve where $\theta=\dfrac{\pi}{4}$.

Ans. 14.6.

CHAPTER XXIII

AREAS OF SURFACES OF REVOLUTION

200. Let $y=f(x)$ be an equation in which $f(x)$ is single valued and continuous for all values of x between and including two values a and b. To find the area of as much of the surface generated by the revolution of the curve $y=f(x)$ about the x-axis as is contained between planes perpendicular to the x-axis through the points whose abscissas are a and b respectively.

Suppose that $f(x)$ is positive for all values of x between a and b.

Under this supposition the curve $y=f(x)$ may be as in Fig. 78.

Let OA and OB represent the abscissas a and b respectively. Divide AB into n equal parts. Call each part Δx. Denote

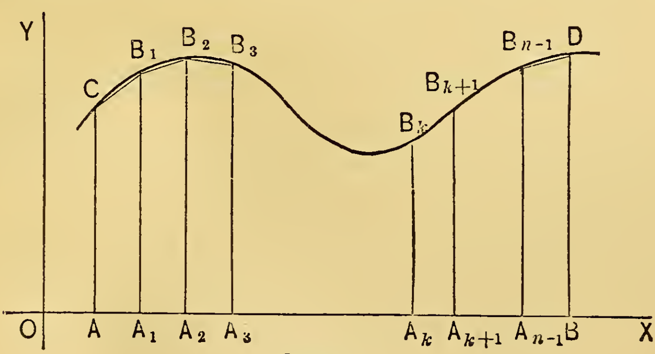


FIG. 78.

the successive points of division of AB by $A_1, A_2, A_3, \cdots, A_{n-1}$. At $A, A_1, A_2, A_3, \cdots, A_{n-1}, B$, erect ordinates to meet the curve in $C, B_1, B_2, B_3, \cdots, B_{n-1}, D$ respectively. Join CB_1, B_1B_2, $B_2B_3, \cdots, B_{n-1}D$.

Let $x_1, x_2, x_3, \cdots, x_{n-1}$, denote the abscissas $OA_1, OA_2, OA_3, \cdots, OA_{n-1}$ respectively.

Since the sum of the chords $CB_1, B_1B_2, B_2B_3, \cdots, B_{n-1}D$, approaches the length of the arc as its limit as the number of chords increases without limit, we should expect that the sum of the areas of the surfaces generated by the chords would approach the area of the surface generated by the arc as its limit as the number of chords increases without limit. On the assumption that such is the case, the area of the surface generated by the arc can be found as follows:

Each of the chords $CB_1, B_1B_2, B_2B_3, \cdots, B_{n-1}D$, generates the frustum of a right circular cone. The curved surface of the frustum of a right circular cone is the product of the slant height and one half the sum of the circumferences of the bases. Therefore the sum of the areas of the surfaces generated by

$$\text{the chords} = \pi[\{2f(a)+\Delta_1 y\}\Delta_1 c + \{2f(x_1)+\Delta_2 y\}\Delta_2 c$$

$$+\{2f(x_2)+\Delta_3 y\}\Delta_3 c + \cdots + \{2f(x_{n-1})+\Delta_n y\}\Delta_n c].$$

Therefore the required area of the surface

$$= \lim_{n=\infty} \pi[\{2f(a)+\Delta_1 y\}\Delta_1 c + \{2f(x_1)+\Delta_2 y\}\Delta_2 c$$

$$+\{2f(x_2)+\Delta_3 y\}\Delta_3 c + \cdots + \{2f(x_{n-1})+\Delta_n y\}\Delta_n c].$$

The area of the surface generated by the chord B_kB_{k+1} is $\pi\{2f(x_k)+\Delta_{k+1}y\}\Delta_{k+1}c$. (2) Compare this area with $2\pi f(x_k)\Delta_{k+1}c$.

Since $\Delta_{k+1}y$ approaches zero as n increases without limit, therefore

$$\lim_{n=\infty}\left[\frac{\pi\{2f(x_k)+\Delta_{k+1}y\}\Delta_{k+1}c}{2\pi f(x_k)\Delta_{k+1}c}\right]=1.$$

Therefore each term in (1), by the theorem of Art. 186, may be replaced by $2\pi f(x_k)\Delta_{k+1}c$ where k has the proper value for that term. Let k take in succession the values $0, 1, 2, 3, \cdots, n$. Denote x_k when $k=0$ by a.

Therefore the required area of the surface

$$= \lim_{n=\infty}\left[2\pi\{f(a)\Delta_1 c + f(x_1)\Delta_2 c + f(x_2)\Delta_3 c + \cdots + f(x_{n-1})\Delta_n c\}\right]$$

$$= \lim_{n=\infty}\sum_{x=a}^{x=b} 2\pi f(x)\Delta c$$

$$= \lim_{n=\infty}\sum_{x=a}^{x=b} 2\pi f(x)\sqrt{\Delta x^2 + \Delta y^2}$$

$$= 2\pi\int_a^b f(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx, \quad \text{or}$$

$$= 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx.$$

The required area of the surface can also be expressed as

$$2\pi\int_c^d y\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy$$

by dividing and multiplying by dy. In this integral, c and d are the values of y, corresponding to the values a and b of x respectively.

201. In the equation $y = f(x)$ of the preceding article, if x is a single-valued function of y between the values c and d of y, and is positive for all values of y between c and d, the area of as much of the surface generated by the revolution of the curve $y = f(x)$ about the y-axis as is contained between planes perpendicular to the y-axis through the points on the curve whose abscissas are a and b respectively can be found in a similar manner to be

$$2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx, \quad \text{or}$$

$$2\pi\int_c^d x\sqrt{1+\left(\frac{dx}{dy}\right)^2}dy.$$

EXERCISES

1. If $f(x)$ of Art. 200 is negative for all values of x between a and b, show that the area of the surface defined in that article is

$$-2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

2. If x of Art. 201 is negative for all values of y between c and d, show that the area of the surface defined in that article is

$$-2\pi\int_a^b x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

202. As illustrations of the application of the above principles to problems in areas of surfaces, consider the following examples:

EXAMPLE 1. Find the area of as much of the surface generated by the revolution about the x-axis of the catenary $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ as is contained between planes perpendicular to the x-axis through the points whose abscissas are 0 and a respectively.

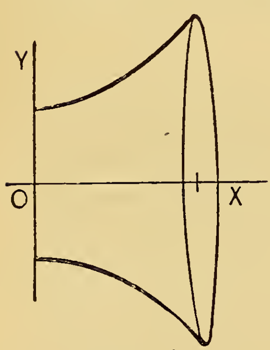


FIG. 79.

The required area of the surface

$$=2\pi\int_0^a y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=2\pi\int_0^a \frac{a}{4}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)^2 dx$$

$$=\frac{\pi a^2}{4}\left(e^2+4-\frac{1}{e^2}\right).$$

EXAMPLE 2. Find the area of as much of the surface generated by the revolution about the y-axis of the catenary $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ as is contained between planes perpendicular to the y-axis through the points whose abscissas are 0 and a respectively.

The required area of the surface

$$= 2\pi\int_0^a x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \pi\int_0^a x\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)dx$$

$$= \pi\left[axe^{\frac{x}{a}} - a^2e^{\frac{x}{a}} - axe^{-\frac{x}{a}} - a^2e^{-\frac{x}{a}}\right]_0^a,$$

by integration by parts,

$$= 2\pi a^2\left(1-\frac{1}{e}\right).$$

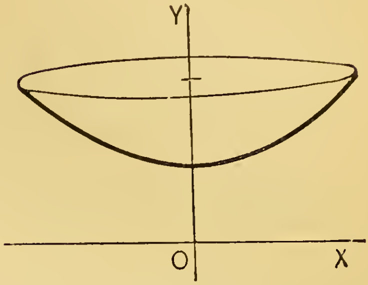


FIG. 80.

EXERCISES

1. Find the area of as much of the surface generated by the revolution about the x-axis of the parabola $y^2 = 4ax$ as is contained between the origin and the plane $x = a$.

Ans. $\frac{8}{3}(\sqrt{8}-1)\pi a^2$.

2. Find the area of the surface of the torus formed by the revolution of the curve $x^2+(y-b)^2 = a^2$, $b > a$, about the x-axis. *Ans.* $4\pi^2 ab$.

3. Find the area of the smaller surface formed by the revolution of the curve $x^2+(y-b)^2 = a^2$, $b < a$, about the x-axis.

Ans. $4\pi a\left[\sqrt{a^2-b^2} - b\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right]$.

4. Find the area of the surface generated by the revolution of the four cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis.

Ans. $\frac{12}{5}\pi a^2$.

5. The axis of a parabolic reflector is 20 inches long and the focus is $4\frac{1}{2}$ inches from the vertex. Find the area of the surface of the reflector. *Ans.* 686π sq. in.

6. Find the area of the surface generated by the revolution of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ about the x-axis. *Ans.* $\frac{64}{3}\pi a^2$.

CHAPTER XXIV

VOLUMES BY MEANS OF PARALLEL CROSS SECTIONS

203. If the areas of the cross sections of a solid made by a set of parallel planes, drawn in any convenient manner, can be expressed in terms of the distances of these planes from a fixed point, the volume of the solid can readily be determined. The following example will illustrate the method.

Example. Find the volume of the right circular cone of height h and radius of base a.

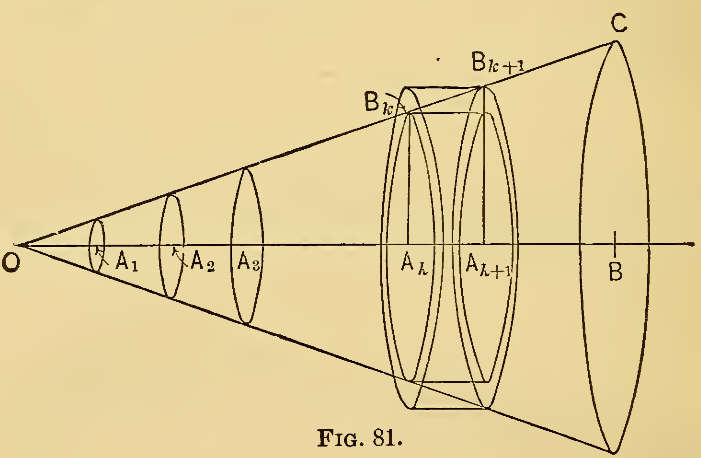


Fig. 81.

Take the vertex of the cone as the origin, and the axis of the cone as the x-axis (see Fig. 81). Let B denote the point at which the x-axis cuts the base of the cone. Divide OB into n equal parts. Call each part Δx. Denote the successive points of division of OB by A_1, A_2, A_3, $\cdots$. Through each of these points pass a plane perpendicular to the x-axis. Let the planes through A_k and A_{k+1} intersect OC in B_k and B_{k+1} respectively. On the circles of intersection of the planes

through A_k and A_{k+1} with the cone construct right cylinders of height Δx.

Denote the abscissas of $A_1, A_2, A_3, \cdots$ by $x_1, x_2, x_3, \cdots$ respectively. Denote the radii of the circles of intersection of the planes through A_k and A_{k+1} with the cone by y_k and y_{k+1} respectively.

From similar triangles,

$$OA_k : A_kB_k :: OB : BC.$$

$$\therefore\ x_k : y_k :: h : a.$$

$$\therefore\ y_k = \frac{ax_k}{h}.$$

Similarly, $$y_{k+1} = \frac{a(x_k + \Delta x)}{h}.$$

Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$, denote the elements of volume into which the cone is divided by the planes through $A_1, A_2, A_3, \cdots$, perpendicular to the x-axis.

The volumes of the right circular cylinders, the radii of whose bases are y_k and y_{k+1}, are $\frac{\pi a^2 x_k^2}{h^2}\Delta x$ and $\frac{\pi a^2 (x_k+\Delta x)^2}{h^2}\Delta x$ respectively.

$$\therefore\ \frac{\pi a^2 x_k^2}{h^2}\Delta x < \alpha_{k+1} < \frac{\pi a^2(x_k+\Delta x)^2}{h^2}\Delta x.$$

Divide by $\frac{\pi a^2 x_k^2}{h^2}\Delta x$.

$$\therefore\ 1 < \frac{\alpha_{k+1}}{\frac{\pi a^2 x_k^2 \Delta x}{h^2}} < \frac{(x_k+\Delta x)^2}{x_k^2}.$$

As $\Delta x \doteq 0$, or when $n = \infty$, limit $\left[\frac{(x_k+\Delta x)^2}{x_k^2}\right] = 1.$

$$\therefore\ \lim_{n=\infty}\left[\frac{\alpha_{k+1}}{\frac{\pi a^2 x_k^2 \Delta x}{h^2}}\right] = 1.$$

Therefore, by the theorem of Art. 186, α_{k+1} may be replaced by $\dfrac{\pi a^2 x_k{}^2 \Delta x}{h^2}$ in any problem involving the limit of the sum of the infinitesimals α.

Now the volume of the cone

$$= [\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n]$$

$$= \underset{n=\infty}{\text{limit}} \left[\alpha_1 + \alpha_2 + \alpha_3 + \cdots + \alpha_n\right]. \tag{1}$$

Let α_{k+1} take in succession the values $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$, and x_k the values $0, x_1, x_2, \cdots, x_{n-1}$. Substitute in (1).

Therefore the volume of the cone

$$= \underset{n=\infty}{\text{limit}} \left[\frac{\pi a^2}{h^2}\left\{0^2 + x_1{}^2 + x_2{}^2 + \cdots + x^2_{n-1}\right\}\Delta x\right]$$

$$= \underset{n=\infty}{\text{limit}} \sum_{x=0}^{x=h} \frac{\pi a^2 x^2}{h^2}\Delta x$$

$$= \frac{\pi a^2}{h^2}\int_0^h x^2 dx$$

$$= \tfrac{1}{3}\pi a^2 h.$$

204. Of the solids, the areas of cross sections of which made by a set of parallel planes can be expressed in terms of the distances of these planes from a fixed point, an important class is those formed by revolving a plane curve about a straight line in its plane.

Let $y = f(x)$ be an equation in which $f(x)$ is single valued and continuous for all values of x between and including two values a and b. To find the volume of as much of the solid generated by revolving the curve $y = f(x)$ about the x-axis as is contained between planes perpendicular to the x-axis through the points whose abscissas are a and b respectively.

Suppose that the curve is as in Fig. 82.

Let OA and OB represent the abscissas a and b respectively. Divide AB into n equal parts. Call each part Δx. Denote

the successive points of division of AB by A_1, A_2, A_3, $\cdots$, A_{n-1}. At each of these points draw the ordinate to the curve. Draw B_kC_{k+1} and $B_{k+1}D_k$ parallel to the x-axis.

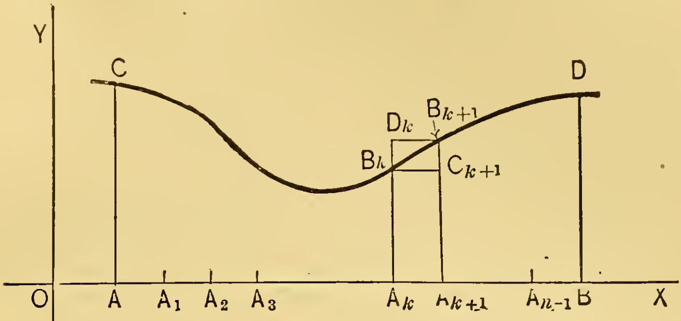


FIG. 82.

Denote the abscissas of A_1, A_2, A_3, $\cdots$, A_{n-1}, by $x_1\, x_2\, x_3 \cdots$, x_{n-1} respectively.

Let α_1, α_2, α_3, $\cdots$, α_n, denote the elements of volume into which the given solid is divided by the planes generated by the revolution about the x-axis of the ordinates from A, A_1, A_2, A_3, $\cdots$, A_{n-1}, B.

$$\therefore\ \pi\{f(x_k)\}^2\Delta x < \alpha_{k+1} < \pi\{f(x_k+\Delta x)\}^2\Delta x.$$

Divide by $\pi\{f(x_k)\}^2\Delta x$.

$$\therefore\ 1 < \frac{\alpha_{k+1}}{\pi\{f(x_k)\}^2\Delta x} < \frac{\{f(x_k+\Delta x)\}^2}{\{f(x_k)\}^2}.$$

As $\Delta x \doteq 0$, or when $n=\infty$, limit $\left[\dfrac{\{f(x_k+\Delta x)\}^2}{\{f(x_k)\}^2}\right]=1.$

$$\therefore\ \underset{n=\infty}{\text{limit}}\left[\frac{\alpha_{k+1}}{\pi\{f(x_k)\}^2\Delta x}\right]=1.$$

Therefore, by the theorem of Art. 186, α_{k+1} may be replaced by $\pi\{f(x_k)\}^2\Delta x$ in any problem involving the limit of the sum of the infinitesimals α.

Now the required volume $=[\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n]$

$$= \underset{n=\infty}{\text{limit}}\left[\alpha_1+\alpha_2+\alpha_3+\cdots+\alpha_n\right]. \quad (1)$$

Let α_{k+1} take in succession the values $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$, and x_k the values $a, x_1, x_2, x_3, \cdots, x_{n-1}$. Substitute in (1).

Therefore the required volume

$$= \lim_{n=\infty} \Big[\pi\Big(\{f(a)\}^2 + \{f(x_1)\}^2$$

$$+ \{f(x_2)\}^2 + \cdots + \{f(x_{n-1})\}^2\Big)\Delta x\Big]$$

$$= \lim_{n=\infty} \sum_{x=a}^{x=b} \pi\{f(x)\}^2 \Delta x$$

$$= \pi\int_a^b \{f(x)\}^2 dx, \text{ or}$$

$$= \pi\int_a^b y^2 dx.$$

205. In the equation $y = f(x)$ of the preceding article, if x is a single-valued function of y, the volume of as much of the solid generated by the revolution of the curve $y = f(x)$ about the y-axis, as is contained between planes perpendicular to the y-axis through the points on the curve whose ordinates are c and d respectively can readily be found to be

$$\pi\int_c^d x^2 dy.$$

206. Example 1. Find the volume of the solid generated by the revolution about the x-axis of as much of the parabola $y^2 = 2\,mx$ as is contained between the origin and the plane perpendicular to the x-axis through the point whose abscissa is $2\,m$.

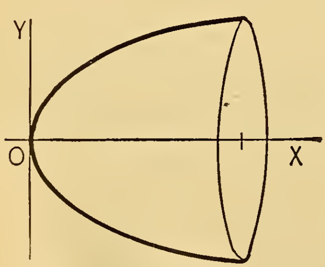


Fig. 83.

The required volume

$$= \lim_{n=\infty} \sum_{x=0}^{x=2m} 2\,\pi m x\,dx$$

$$= 2\,\pi m\int_0^{2m} x\,dx$$

$$= 4\,\pi m^3.$$

EXAMPLE 2. Find the volume of the solid generated by the revolution about the y-axis of as much of the parabola $y^2=2mx$ as is contained between the origin and the plane perpendicular to the y-axis through the point whose abscissa is $2m$.

The required volume

$$= \pi\int_0^{2m}\left(\frac{y^2}{2m}\right)^2 dy$$

$$= \frac{\pi}{4m^2}\int_0^{2m} y^4 dy$$

$$= \tfrac{8}{5}\pi m^3.$$

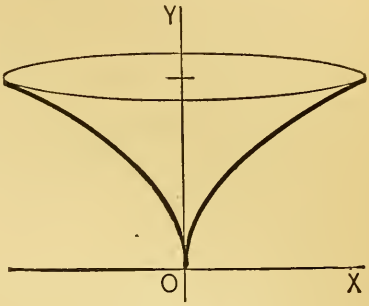


FIG. 84.

EXERCISES

1. A tree is 24 inches in diameter. A notch is cut to the center of the tree, one face being horizontal and the other inclined 45° to the horizontal. Find the amount of wood cut out. *Ans.* $\frac{2}{3}$ cu. ft.

2. A variable equilateral triangle moves with its plane perpendicular to the x-axis and the ends of its base on the parts of the curves $y^2=16ax$ and $y^2=4ax$ respectively above the x-axis. Find the volume generated by the triangle as it moves from the origin to the points whose abscissa is a. *Ans.* $\frac{\sqrt{3}}{2}a^3$.

3. The cap of a banister post is a solid, any cross section of which is a square. A vertical plane through the diagonal of the base contains one diagonal of any such square, and the ends of this diagonal lie on a circle of radius 8 inches with its center 4 inches above the plane of the base. Find the volume of the cap. *Ans.* 1152 cu. in.

4. Find the volume of the torus formed by the revolution of the curve $x^2+(y-b)^2=a^2$, $b>a$ about the x-axis.

Ans. $2\pi^2a^2b$.

5. Find the volume of the circular spindle formed by the revolution about the x-axis of the part of the curve $x^2+(y-b)^2=a^2$, $b<a$.

Ans. $2\pi\left[\frac{b^2+2a^2}{3}\sqrt{a^2-b^2}-a^2b\cos^{-1}\frac{b}{a}\right]$.

6. Find the volume of as much of the solid generated by the revolution of the curve $y=a\sin^{-1}\frac{x}{a}$ about the x-axis as is contained between the origin and the plane perpendicular to the x-axis through the point whose abscissa is a.

Ans. $\frac{\pi a^3}{4}[\pi^2-8]$.

7. A steel band is placed on a cylindrical boiler. A cross section of the band is a semi-ellipse whose semi-axes are 3 and $\frac{3}{\sqrt{6}}$ inches respectively, the greater being parallel to the axis of the boiler. The diameter of the boiler is 48 inches. Find the volume of the band. *Ans.* $6\pi[1+6\sqrt{6}\,\pi]$ cu. in.

8. Find the ratio between the volume of the spindle formed by the revolution of a parabola about an ordinate and the volume of the circumscribing cylinder. *Ans.* $8:15$.

9. Find the volume of the solid formed by the revolution of one arch of the curve $y=2\sin\frac{x}{10}$ about the x-axis.

Ans. $20\pi^2$.

10. A bead is of the form of a sphere through which a round hole has been bored, the axis of the hole coinciding with a diameter of the sphere. If the diameter of the hole is two thirds of the radius of the sphere, compare the volume bored out with the volume of the sphere. *Ans.* $27-16\sqrt{2}:27$.

CHAPTER XXV

SUCCESSIVE INTEGRATION

207. In finding the integral of a function of two or more independent variables, we hold all the variables constant except the one with respect to which we are performing the integration, and treat the function as if it were a function of one variable alone.

For example, in finding $\int x^2y^2\,dx$, we hold y constant, and treat x^2y^2 as if it were a function of x alone.

Thus, $$\int x^2y^2\,dx = \frac{x^3y^2}{3} + c.$$

This is done only when the variables are independent. We saw in finding the lengths of arcs of curves, in the case of two variables when one was dependent, that we had to express one in terms of the other, or both in terms of a third variable before the integration could be performed.

208. Let $f(x, y)$ be a function of two independent variables.

Definitions. $\int f(x, y)dx$, or $\int f(x, y)dy$, or $\int f(x, y)dz$ where z is any third variable, is called a **single integral** of $f(x, y)$.

An integral of any single integral of $f(x, y)$, with respect to any variable, is called a **double integral** of $f(x, y)$.

Thus, $\int \left\{ \int f(x, y)dx \right\} dy$ is a double integral of $f(x, y)$.

An integral of any double integral of $f(x, y)$, with respect to any variable, is called a **triple integral** of $f(x, y)$.

Thus, $\int\left[\int\left\{\int f(x,\ y)\,dx\right\}dy\right]dz$ is a triple integral of $f(x,\ y)$.

These definitions are given for definiteness for a function of two independent variables. They would be similar, however, for a function of one variable or of any number of independent variables.

The process of taking integrals in succession of any given function is called **successive integration.**

209. The order in which the integration is to be performed may be indicated by brackets as in the preceding illustrations, or by any other convenient notation. We shall employ the notation most commonly adopted in text-books, which is to write the integral without brackets, and arrange the differentials so that their order, reading *from right to left*, will indicate the order in which the integration is to be performed.

Thus, $\int\left\{\int\left(\int dz\right)dy\right\}dx$ will be written $\int\int\int dx\,dy\,dz.$

EXERCISES

Show that:

1. $\displaystyle\int_0^a\int_0^b (x^2+y^2)\,dx\,dy = \frac{ab}{3}(a^2+b^2).$

2. $\displaystyle\int_0^{2a}\int_0^x (x^2+y^2)\,dx\,dy = \tfrac{16}{3}\,a^4.$

3. $\displaystyle\int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta\,d\theta\,dr = \tfrac{1}{3}\,a^2.$

4. $\displaystyle\int_{-a}^{a}\int_0^{\frac{y^2}{a}} (x+y)\,dy\,dx = \tfrac{1}{5}\,a^3.$

5. $\displaystyle\int_b^c\int_0^x\int_0^{\frac{y^2}{2}} (x+y)\,dx\,dy\,dz = \tfrac{7}{120}(c^5-b^5).$

6. $\displaystyle\int_{-a}^{a}\int_{-x}^{2x}\int_0^z (x+y+z)\,dx\,dy\,dz = 3\,a^3z.$

CHAPTER XXVI

PLANE AREAS BY DOUBLE INTEGRATION

RECTANGULAR COÖRDINATES

210. Let $y = f(x)$ be an equation in which $f(x)$ is single valued and continuous for all values of x between and including two values a and b. To find by double integration the area inclosed by the curve $y = f(x)$, the x-axis, and the ordinates corresponding to the abscissas a and b respectively.

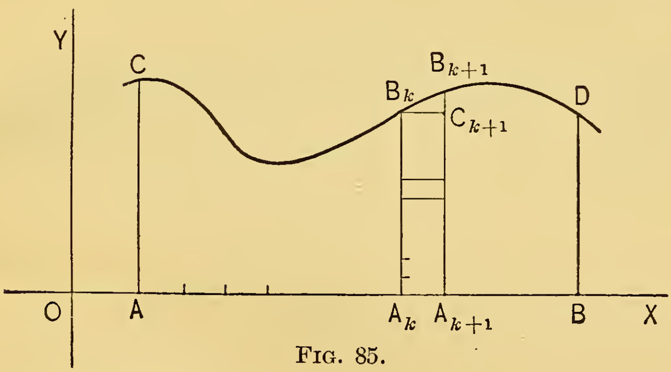


FIG. 85.

Suppose that $f(x)$ is positive for all values of x between a and b. Under this supposition, the curve $y = f(x)$ may be as in Fig. 85.

Let OA and OB represent the abscissas a and b respectively. Divide AB into n equal parts. Call each part Δx. At A_k and A_{k+1}, two consecutive points of division, erect ordinates to meet the curve in B_k and B_{k+1} respectively. Divide A_kB_k into m equal parts. Call each part Δy. From each of the points of division of A_kB_k, draw a line parallel to the x-axis to meet $A_{k+1}B_{k+1}$.

Denote the abscissa OA_k by x_k.

The area of each of the small rectangles is $\Delta y\,\Delta x$. By summing the Δy's we get the area of the rectangle $A_kB_kC_{k+1}A_{k+1}$ to be $\left(\sum_{y=0}^{y=f(x_k)} \Delta y\right)\Delta x$, and therefore $\underset{m=\infty}{\text{limit}}\left(\sum_{y=0}^{y=f(x_k)} \Delta y\right)\Delta x$, or $\left(\int_0^{f(x_k)} dy\right)\Delta x$.

Let x_k take in succession the values $a, x_1, x_2, \cdots, x_{n-1}$. Therefore the sum of the areas of the rectangles that may be described on the n equal parts of division of the line AB

$$= \left[\int_0^{f(a)} dy + \int_0^{f(x_1)} dy + \int_0^{f(x_2)} dy + \cdots + \int_0^{f(x_{n-1})} dy\right]\Delta x$$

$$= \sum_{x=a}^{x=b}\left(\int_0^{f(x)} dy\right)\Delta x.$$

In Chapter XIX it was seen that the limit of the sum of these areas is the required area.

$$\text{Therefore the required area} = \underset{n=\infty}{\text{limit}}\left[\sum_{x=a}^{x=b}\left(\int_0^{f(x)} dy\right)\Delta x\right]$$

$$= \int_a^b\left(\int_0^{f(x)} dy\right)dx$$

$$= \int_a^b \int_0^{f(x)} dx\,dy.$$

211. Example. Find by double integration the area of the circle $x^2 + y^2 = a^2$.

Since $y = \pm\sqrt{a^2 - x^2}$, y is a double-valued function of x. This difficulty can be avoided by making use of the fact that the area of the circle is four times the area in the first quadrant. We shall therefore drop the minus sign and confine our attention to the first quadrant.

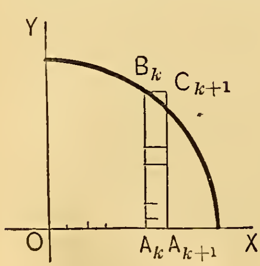


Fig. 86.

Make the same construction as in the general case.

The area of the rectangle $A_kC_{k+1} = \underset{m=\infty}{\text{limit}}\left(\sum_{y=0}^{y=\sqrt{a^2-x^2}} \Delta y\right)\Delta x$

$$= \left(\int_0^{\sqrt{a^2-x^2}} dy\right)\Delta x.$$

Let x_k take in succession the values $0, x_1, x_2, \cdots, x_{n-1}$. Therefore the area of the circle

$$= 4 \underset{n=\infty}{\text{limit}}\left[\sum_{x=0}^{x=a}\left(\int_0^{\sqrt{a^2-x^2}} dy\right)\Delta x\right]$$

$$= 4\int_0^a\left(\int_0^{\sqrt{a^2-x^2}} dy\right)dx$$

$$= 4\int_0^a\int_0^{\sqrt{a^2-x^2}} dx\,dy$$

$$= \pi a^2.$$

EXERCISE

If $f(x)$ of Art. 210 is negative for all values of x between a and b, show that the area defined in that article is $-\int_a^b\int_0^{f(x)} dx\,dy$.

POLAR COÖRDINATES

212. Let $r=f(\theta)$ be an equation in which $f(\theta)$ is single valued and continuous for all values of θ between and including two values α and β. To find, by double integration, the area inclosed by the curve $r=f(\theta)$, and the radii vectores that make angles of α and β respectively with the initial line.

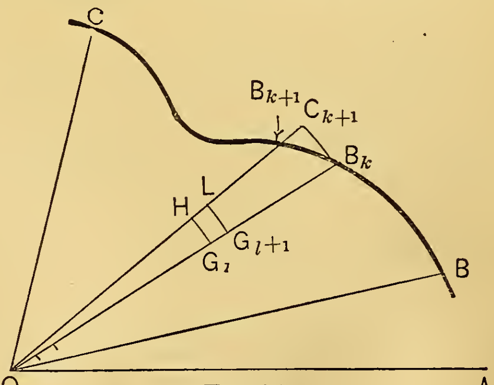


FIG. 87.

Suppose that the curve $r=f(\theta)$ is as in Fig. 87.

Let AOB and AOC denote the angles α and β respectively. Divide the

angle BOC into n equal parts. Call each part $\Delta\theta$. Denote the points at which two consecutive lines of division of the angle BOC meet the curve by B_k and B_{k+1}. Divide OB_k into m equal parts. Call each part Δr. Denote two consecutive points of division of OB_k by G_l and G_{l+1}. Denote the distances of these points from O by r_l and r_{l+1} respectively. With center O and radii r_l and r_{l+1} describe arcs of circles to meet OB_{k+1} in H and L respectively.

Denote the angle AOB_k by θ_k.

$$\text{The area of } G_lG_{l+1}LH = \text{the area of } OG_{l+1}L - \text{the area of } OG_lH$$

$$= \tfrac{1}{2}[(r_l + \Delta r)^2 - r_l^2]\Delta\theta$$

$$= [r_l + \tfrac{1}{2}\Delta r]\Delta r\Delta\theta.$$

By giving r_l in succession the values $0, r_1, r_2, \cdots, r_{m-1}$, and summing the results, we get the sum of the areas of all the figures formed by the radii OB_k and OB_{k+1}, and the arcs of the circles. That is, we get the area of the sector OB_kC_{k+1}. Therefore the area of the sector OB_kC_{k+1}

$$= \{(0 + \tfrac{1}{2}\Delta r) + (r_1 + \tfrac{1}{2}\Delta r) + (r_2 + \tfrac{1}{2}\Delta r) + \cdots$$
$$+ (r_{m-1} + \tfrac{1}{2}\Delta r)\}\Delta r\,\Delta\theta$$

$$= \underset{m=\infty}{\text{limit}}\Big[\{(0 + \tfrac{1}{2}\Delta r) + (r_1 + \tfrac{1}{2}\Delta r) + (r_2 + \tfrac{1}{2}\Delta r) + \cdots$$
$$+ (r_{m-1} + \tfrac{1}{2}\Delta r)\}\Delta r\,\Delta\theta\Big]. \tag{1}$$

Now $\underset{m=\infty}{\text{limit}}\left[\dfrac{(r_l + \frac{1}{2}\Delta r)\Delta r\,\Delta\theta}{r_l\Delta r\,\Delta\theta}\right] = 1$. Therefore, by the theorem of Art. 186, each term in (1) may be replaced by $r_l\Delta r\,\Delta\theta$ where r_l has the proper value for that term. Therefore the area of

$$\text{the sector} = \underset{m=\infty}{\text{limit}}\Big[\{0 + r_1 + r_2 + \cdots + r_{m-1}\}\Delta r\,\Delta\theta\Big]$$

$$= \left(\underset{m=\infty}{\text{limit}}\sum_{r=0}^{r=f(\theta_k)} r\,\Delta r\right)\Delta\theta = \left(\int_0^{f(\theta_k)} r\,dr\right)\Delta\theta.$$

Let θ_k take in succession the values $\alpha, \theta_1, \theta_2, \cdots, \theta_{n-1}$. Therefore the sum of the areas of all the circular sectors

$$=\left\{\int_0^{f(\alpha)} r\,dr+\int_0^{f(\theta_1)} r\,dr+\int_0^{f(\theta_2)} r\,dr+\cdots+\int_0^{f(\theta_{n-1})} r\,dr\right\}\Delta\theta$$

$$=\sum_{\theta=\alpha}^{\theta=\beta}\left(\int_0^{f(\theta)} r\,dr\right)\Delta\theta.$$

In Chapter XX, it was seen that the limit of the sum of these areas is the required area.

Therefore the required area $=\underset{n=\infty}{\text{limit}}\left[\sum_{=\alpha}^{\theta=\beta}\left(\int_0^{f(\theta)} r\,dr\right)\Delta\theta\right]$

$$=\int_\alpha^\beta\left(\int_0^{f(\theta)} r\,dr\right)d\theta$$

$$=\int_\alpha^\beta\int_0^{f(\theta)} r\,d\theta\,dr.$$

213. Example. Find by double integration the area of the cardioide $r=2a(1-\cos\theta)$.

Since the curve is symmetrical to the initial line, we may consider the part above the initial line and multiply the result by 2.

Make the same construction as in the general case.

The area of the sector

$$OB_kC_{k+1}=\left(\underset{m=\infty}{\text{limit}}\sum_{r=0}^{r=2a(1-\cos\theta_k)} r\,\Delta r\right)\Delta\theta$$

$$=\left(\int_0^{2a(1-\cos\theta_k)} r\,dr\right)\Delta\theta.$$

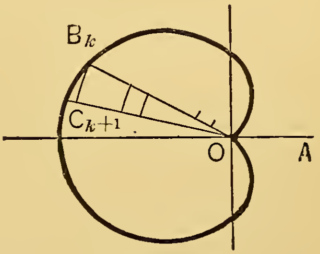


Fig. 88.

Therefore the area of the cardioide

$$=2\left\{\underset{n=\infty}{\text{limit}}\sum_{\theta=0}^{\theta=\pi}\left(\int_0^{2a(1-\cos\theta)} r\,dr\right)\Delta\theta\right\}$$

$$=2\int_0^\pi\int_0^{2a(1-\cos\theta)} r\,d\theta\,dr$$

$$=6\pi a^2.$$

EXERCISES

1. A comet, C, travels around the sun, S, in a parabola, S being the focus. Compare the areas swept out by SC as θ moves from 160° to 150°; from 90° to 80°; from 10° to 0°, θ being the angle CSO, where O is the vertex of the parabola.

Ans. 4541 : 29.7 : 8.77.

2. Solve Exercises 23 and 24, Chapter XIX, by double integration.

3. Solve Exercises 4 and 6, Chapter XX, by double integration.

4. In each of the remaining exercises of Chapters XIX and XX, express the area in terms of a double integral. Do not perform the integration.

CHAPTER XXVII

VOLUMES OF REVOLUTION BY DOUBLE INTEGRATION

RECTANGULAR COÖRDINATES

214. Let $y = f(x)$ be an equation in which $f(x)$ is single valued and continuous for all values of x between and including two values a and b. To find by double integration the volume of as much of the solid generated by revolving the curve $y = f(x)$ about the x-axis as is contained between planes perpendicular to the x-axis through points whose abscissas are a and b respectively.

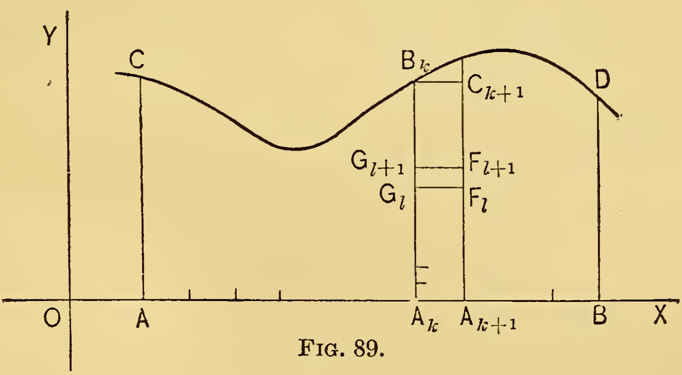


Fig. 89.

Make the same construction as in Fig. 85, Art. 210 (see Fig. 89). Denote two consecutive points of division of A_kB_k by G_l and G_{l+1}. Denote the ordinate of G_l by y_l.

The volume generated by the element of area $G_lG_{l+1}F_{l+1}F_l$

$$= \pi\left[(y_l + \Delta y)^2 - y_l^2\right]\Delta x$$

$$= 2\pi\left[y_l + \tfrac{1}{2}\Delta y\right]\Delta y\,\Delta x.$$

By giving y_l in succession the values $0, y_1, y_2, y_3, \cdots, y_{m-1}$, and summing the results, we get that the volume generated by the rectangle $A_kB_kC_{k+1}A_{k+1}$

$$= 2\pi[(0+\tfrac{1}{2}\Delta y)+(y_1+\tfrac{1}{2}\Delta y)+(y_2+\tfrac{1}{2}\Delta y)+\cdots + (y_{m-1}+\tfrac{1}{2}\Delta y)]\,\Delta y\,\Delta x$$

$$= \underset{m=\infty}{\text{limit}}\left[2\pi\{(0+\tfrac{1}{2}\Delta y)+(y_1+\tfrac{1}{2}\Delta y)+(y_2+\tfrac{1}{2}\Delta y)+\cdots + (y_{m-1}+\tfrac{1}{2}\Delta y)\}\,\Delta y\,\Delta x\right]. \qquad (1)$$

Now $\underset{m=\infty}{\text{limit}}\left[\dfrac{y_l+\frac{1}{2}\Delta y}{y_l}\right]=1$. Therefore, by the theorem of Art. 186, each term in (1) may be replaced by y_l, where l has the proper value for that term.

Therefore the volume generated by the rectangle $A_kB_kC_{k+1}A_{k+1}$

$$= \underset{m=\infty}{\text{limit}}\left[2\pi\{0+y_1+y_2+\cdots+y_{m-1}\}\,\Delta y\,\Delta x\right]$$

$$= \left[\underset{m=\infty}{\text{limit}}\sum_{y=0}^{y=f(x_k)} 2\pi y\,\Delta y\right]\Delta x$$

$$= 2\pi\left(\int_0^{f(x_k)} y\,dy\right)\Delta x.$$

Let x_k take in succession the values $a, x_1, x_2, \cdots, x_{n-1}$. Therefore the sum of the volumes generated by all the rectangles described on the n equal parts of division of the line AB

$$= \sum_{x=a}^{x=b}\left(2\pi\int_0^{f(x_k)} y\,dy\right)\Delta x.$$

In Art. 204, it was shown that the limit which this sum approaches is the required volume.

Therefore the required volume

$$= \underset{n=\infty}{\text{limit}}\left[\sum_{x=a}^{x=b}\left(2\pi\int_0^{f(x)} y\,dy\right)\Delta x\right]$$

$$= \int_a^b\left(2\pi\int_0^{f(x)} y\,dy\right)dx$$

$$= 2\pi\int_a^b\int_0^{f(x)} y\,dx\,dy.$$

In the equation $y = f(x)$, if x is a single-valued function of y, the volume of as much of the solid generated by the revolution of the curve $y = f(x)$ about the y-axis, as is contained between planes perpendicular to the y-axis through the points on the curve whose ordinates are c and d respectively can be found in a similar manner to be

$$= 2\pi \int_c^d \int_0^x x\,dy\,dx.$$

215. EXAMPLE. Find, by double integration, the volume generated by the revolution about the x-axis of as much of the parabola $y^2 = 2x$ as is contained between the origin and the plane perpendicular to the x-axis through a point whose abscissa is 4.

The curve is as in Fig. 90.

Make the same construction as in Fig. 85, Art. 210.

The volume of the cylinder generated by the revolution of the rectangle $A_k C_{k+1}$

$$= (2\pi \int_0^{\sqrt{2x}} y\,dy)\Delta x.$$

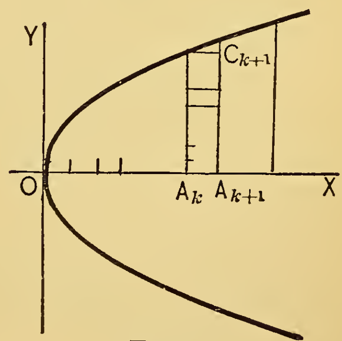


FIG. 90.

Therefore the required volume

$$= 2\pi \int_0^4 \int_0^{\sqrt{2x}} y\,dx\,dy$$

$$= 2\pi \int_0^4 x\,dx$$

$$= 16\pi.$$

POLAR COÖRDINATES

216. Let $r = f(\theta)$ be an equation in which $f(\theta)$ is single valued and continuous for all values of θ between and including two values α and β. To find the volume generated by the

revolution about the initial line of the radii vectores that make angles of α and β respectively with the initial line, and the part of the curve included between these radii vectores.

Suppose that $f(\theta)$ is positive for all values of θ between α and β.

Under this supposition, the curve may be as in Fig. 91.

Make the same construction as in Fig. 87, Art. 212.

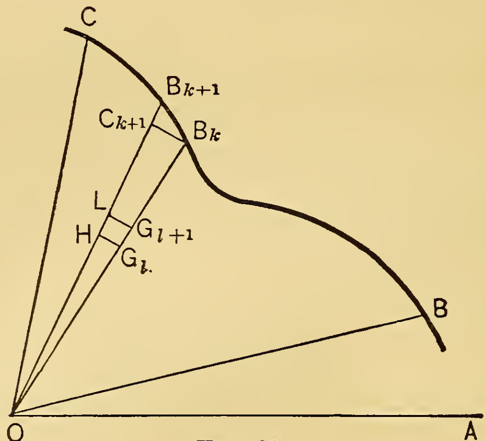


FIG. 91.

The volume generated by revolving the area $G_lG_{l+1}LH$ about the initial line is greater than $2\pi r_l \sin\theta_k[r_l\,\Delta r\,\Delta\theta + \frac{1}{2}\Delta r^2\,\Delta\theta]$, and less than $2\pi(r_l+\Delta r)\sin(\theta_k+\Delta\theta)[r_l\,\Delta r\,\Delta\theta + \frac{1}{2}\Delta r^2\,\Delta\theta]$. Call it $\phi(r_l, \theta_k)\Delta r\,\Delta\theta$.

By giving r_l in succession the values $0, r_1, r_2, \cdots, r_{m-1}$, and summing the results, we get that the volume generated by the sector OB_kC_{k+1}

$$= \left\{ \sum_{r=0}^{r=f(\theta_k)} \phi(r, \theta_k)\Delta r \right\} \Delta\theta$$

$$= \left\{ \underset{m=\infty}{\text{limit}} \sum_{r=0}^{r=f(\theta_k)} \phi(r, \theta_k)\Delta r \right\} \Delta\theta$$

$$= \left(\int_0^{f(\theta_k)} \phi(r, \theta_k)dr \right) \Delta\theta.$$

Let θ_k take in succession the values α, θ_1, θ_2, θ_3, $\cdots$, θ_{n-1}. Therefore the sum of the volumes generated by the circular sectors that may be described on the n equal parts of division of the angle BOC

$$=\sum_{\theta=\alpha}^{\theta=\beta}\left(\int_0^{f(\theta)}\phi(r,\,\theta)\,dr\right)\Delta\theta.$$

Since the sum of these circular sectors approaches the given area as its limit as n increases without limit, the sum of the volumes generated by these sectors approaches the required volume as its limit as n increases without limit.

Therefore the required volume

$$=\lim_{n=\infty}\left[\sum_{\theta=\alpha}^{\theta=\beta}\left(\int_0^{f(\theta)}\phi(r,\,\theta)\,dr\right)\Delta\theta\right]$$

$$=\int_\alpha^\beta\left(\int_0^{f(\theta)}\phi(r,\,\theta)\,dr\right)d\theta$$

$$=\int_\alpha^\beta\int_0^{f(\theta)}\phi(r,\,\theta)\,d\theta\,dr.$$

Now $\quad 2\,\pi\,r_l\sin\theta_k[r_l\,\Delta r\,\Delta\theta+\tfrac{1}{2}\,\Delta r^2\,\Delta\theta]$

$$<\phi(r_l,\,\theta_k)\Delta r\,\Delta\theta<2\,\pi(r_l+\Delta r)\sin(\theta_k+\Delta\theta)\,[r_l\,\Delta r\,\Delta\theta+\tfrac{1}{2}\,\Delta r^2\,\Delta\theta].$$

Divide by $2\,\pi\,r_l^2\sin\theta_k\,\Delta r\,\Delta\theta$.

$$\therefore\ \frac{r_l\,\Delta r\,\Delta\theta+\frac{1}{2}\,\Delta r^2\,\Delta\theta}{r_l\,\Delta r\,\Delta\theta}<\frac{\phi(r_l,\,\theta_k)}{2\,\pi\,r_l^2\sin\theta_k}$$

$$<\frac{(r_l+\Delta r)\sin(\theta_k+\Delta\theta)\,[r_l\,\Delta r\,\Delta\theta+\frac{1}{2}\,\Delta r^2\,\Delta\theta]}{r_l^2\sin\theta_k\,\Delta r\,\Delta\theta}.$$

Since the limit of each extreme of this inequality as m and n both become infinite is 1,

$$\therefore\ \lim_{\substack{m=\infty\\ n=\infty}}\left[\frac{\phi(r_l,\,\theta_k)}{2\,\pi\,r_l\sin\theta_k}\right]=1.$$

Therefore, by the theorem of Art. 186, each term in $\int_\alpha^\beta \int_0^{f(\theta)} \phi(r, \theta) d\theta\, dr$ may be replaced by $2\pi r_l^2 \sin\theta_k \Delta\theta\, \Delta r$, where k and l have the proper values for that term.

Therefore the required volume $= 2\pi \int_\alpha^\beta \int_0^{f(\theta)} r^2 \sin\theta\, d\theta\, dr.$

EXERCISE

In $r = f(\theta)$ of Art. 216, if $f(\theta)$ is negative for all values of θ between α and β, show that the area defined in that article is

$$-2\pi \int_\alpha^\beta \int_0^{f(\theta)} r^2 \sin\theta\, d\theta\, dr.$$

217. As an illustration of the application of the theorem of the preceding article to problems in volume, consider the following example.

Example. Find the volume generated by the revolution about the initial line of the cardioide $r = 2a(1 - \cos\theta)$.

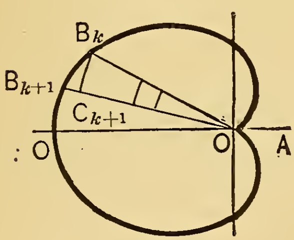


Fig. 92.

The volume of the sector OB_kC_{k+1}

$$= \left(\int_0^{2a(1-\cos\theta_k)} \phi(r, \theta_k) dr\right)\Delta\theta.$$

Therefore the required volume

$$= 2\pi \int_0^\pi \int_0^{2a(1-\cos\theta)} r^2 \sin\theta\, d\theta\, dr$$

$$= \tfrac{64}{3}\pi a^3.$$

EXERCISES

1. Find, by double integration, the volume generated by the revolution of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis; about the y-axis. *Ans.* $\frac{4}{3}\pi ab^2$; $\frac{4}{3}\pi a^2 b$.

2. Find, by double integration, the volume generated by the revolution of the four cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the x-axis. *Ans.* $\frac{32}{105}\pi a^3$.

3. Find, by double integration, the volume generated by the revolution of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about the x-axis; about the y-axis.

Ans. $5\pi^2 a^3$; $6\pi^3 a^3$.

4. Find the volume generated by the revolution of the circle $r = a$ about the initial line. *Ans.* $\frac{4}{3}\pi a^3$.

5. Find the volume generated by the revolution of one loop of the lemniscate $r = a\cos 2\theta$ about the initial line.

Ans. $\dfrac{2\pi a^3}{105}[8\sqrt{2} - 9]$.

6. Find the volume generated by the revolution of one loop of the lemniscate $r = a\sin 2\theta$ about the initial line.

Ans. $\dfrac{32}{105}\pi a^3$.

7. Find the volume generated by the revolution of, (a) the small loop of the curve, (b) the whole curve $r = a\sin\frac{1}{2}\theta$ about the initial line.

Ans. $\dfrac{2\sqrt{2}}{15}\pi a^3$; $\dfrac{2}{15}\left[2^3 - \sqrt{2}\right]\pi a^3$.

8. Find the volume generated by the revolution of, (a) the loop in the first quadrant, (b) the loop below the initial line of the curve $r = a\sin 3\theta$ about the initial line.

Ans. $\dfrac{27\sqrt{3}}{320}\pi a^3$; $\dfrac{27\sqrt{3}}{160}\pi a^3$.

CHAPTER XXVIII

VOLUMES. SURFACES

218. Let $z = f(x, y)$ be the equation of a surface that cuts the xy-plane, the xz-plane, and the planes whose equations are $x = a$ and $x = b$ in curves which together inclose a region on the surface.

Suppose that $f(x, y)$ is single valued and continuous for all values of x and y which together determine a point in the region of the xy-plane bounded by the curve of intersection of the surface and xy-plane, and the lines whose equations are

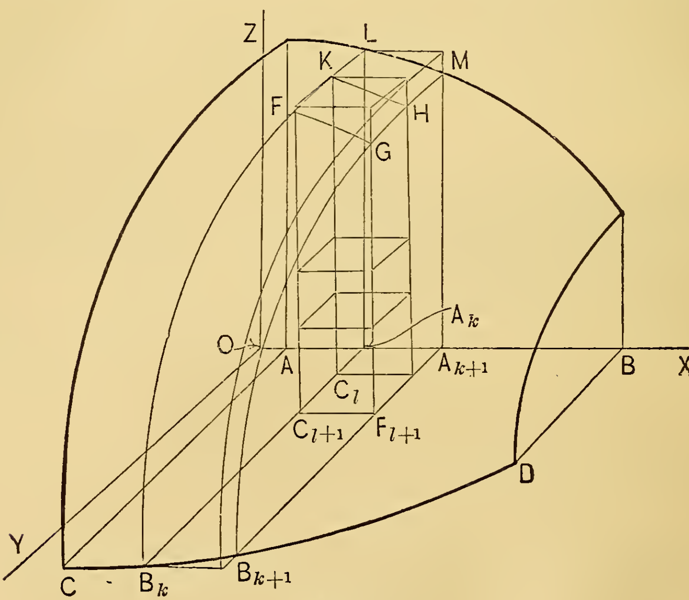


Fig. 93.

$x = a$ and $x = b$. To find the volume inclosed by the xy-plane, the xz-plane, the planes whose equations are $x = a$ and $x = b$, and the given surface.

Suppose that the surface is as in Fig. 93.

Let OA and OB represent a and b respectively. Divide AB into n equal parts. Call each part Δx. At A_k and A_{k+1} draw planes perpendicular to the x-axis to meet the surface in $LKFB_k$ and $MHGB_{k+1}$ respectively. Divide A_kB_k into m equal parts. Call each part Δy. At C_l and C_{l+1} draw planes perpendicular to A_kB_k to meet the surface in KH and FG respectively. Divide C_lK into p equal parts. Call each part Δz. At each of the points of division of C_lK draw a plane perpendicular to C_lK.

Denote the abscissa OA_k by x_k. Denote the ordinate A_kC_l by y_l.

In $z = f(x, y)$, let $z = 0$ and we have the equation of the curve $CB_kB_{k+1}D$. Solve $f(x, y) = 0$ for y. Suppose that the result is $y = \phi(x)$. Then the coördinates of B_k are $(x_k, \phi(x_k), 0)$.

The volume of each of the parallelopipeds formed by the six planes is $\Delta z\, \Delta y\, \Delta x$. By summing the Δz's from C_l to K we get that the volume of the rectangular parallelopiped $F_{l+1}K$

$$= \left(\sum_{z=0}^{z=f(x_k,\, y_l)} \Delta z\right)\Delta y\, \Delta x, \text{ or } \left(\lim_{p=\infty} \sum_{z=0}^{z=f(x_k,\, y_l)} \Delta z\right)\Delta y\, \Delta x,$$

or

$$\left(\int_0^{f(x_k,\, y_l)} dz\right)\Delta y\, \Delta x.$$

This is the volume of a parallelopiped one of whose corners, C_l, is distant y_l from the x-axis. Let y_l take in succession the values $0, y_1, y_2, y_3, \cdots, y_{m-1}$. Then the sum of the volumes of all the rectangular parallelopipeds that can be described on the m equal parts of division of the line A_kB_k is

$$\left[\int_0^{f(x_k,\, 0)} dz + \int_0^{f(x_k,\, y_1)} dz + \int_0^{f(x_k,\, y_2)} dz + \cdots + \int_0^{f(x_k,\, y_{m-1})} dz\right]\Delta y\, \Delta x,$$

or

$$= \left\{\sum_{y=0}^{y=\phi(x_k)} \left(\int_0^{f(x_k,\, y)} dz\right)\Delta y\right\}\Delta x.$$

The limit which this sum approaches as m increases without limit is the volume of the solid bounded by the xy-plane, the xz-plane, the planes whose equations are $x = x_k$ and $x = x_{k+1}$,

and the cylindrical surface one of whose bases is the curve $LKFB_k$.

Then the volume of the solid just described

$$= \left\{ \underset{m=\infty}{\text{limit}} \sum_{y=0}^{y=\phi(x_k)} \left(\int_0^{f(x_k,\, y)} dz \right) \Delta y \right\} \Delta x$$

$$= \left\{ \int_0^{\phi(x_k)} \left(\int_0^{f(x_k,\, y)} dz \right) dy \right\} \Delta x.$$

This is the volume of the solid, one of whose corners, A_k, is at a distance x_k from 0. Let x_k take in succession the values a, x_1, x_2, $\cdots$, x_{n-1}. Then the sum of all the solids that can be described on the n equal parts of division of the line AB is

$$\left\{ \int_0^{\phi(a)} \left(\int_0^{f(a,\, y)} dz \right) dy + \int_0^{\phi(x_1)} \left(\int_0^{f(x_1,\, y)} dz \right) dy + \int_0^{\phi(x_2)} \left(\int_0^{f(x_2,\, y)} dz \right) dy \right.$$
$$\left. + \cdots + \int_0^{\phi(x_{n-1})} \left(\int_0^{f(x_{n-1},\, y)} dz \right) dy \right\} \Delta x,$$

or

$$\sum_{x=a}^{x=b} \left\{ \int_0^{\phi(x)} \left(\int_0^{f(x,\, y)} dz \right) dy \right\} \Delta x.$$

It can be shown by a more or less elaborate investigation which will not be entered into here, that the limit of this sum as n increases without limit is the required volume. Assuming that this limit is the required volume, we have:

$$\text{The required volume} = \underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} \left\{ \int_0^{\phi(x)} \left(\int_0^{f(x,\, y)} dz \right) dy \right\} \Delta x$$

$$= \int_a^b \left\{ \int_0^{\phi(x)} \left(\int_0^{f(x,\, y)} dz \right) dy \right\} dx$$

$$= \int_a^b \int_0^{\phi(x)} \int_0^{f(x,\, y)} dx\, dy\, dz.$$

EXERCISE

Determine the expressions for the volume of Art. 218 when it lies in the other octants.

219. As an illustration of the application of the theorem of the preceding article to problems in volume, consider the following example.

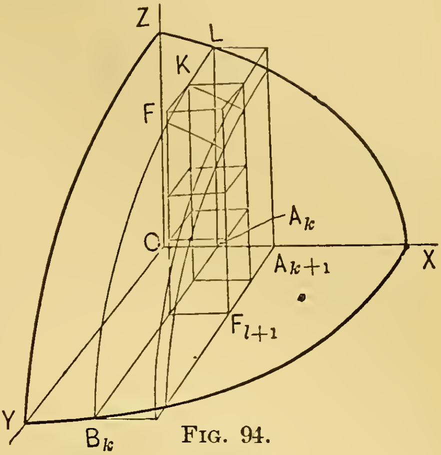


FIG. 94.

EXAMPLE. Find the volume of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Consider the part of the ellipsoid in the first octant.

The surface of this much of the ellipsoid is as in Fig. 94.

Make the same construction as in the general case.

The volume of the parallelopiped $F_{l+1}K$

$$=\left(\underset{p=\infty}{\text{limit}}\sum_{z=0}^{z=c\sqrt{1-\frac{x_k^2}{a^2}-\frac{y_l^2}{b^2}}}\Delta z\right)\Delta y\,\Delta x=\left(\int_0^{c\sqrt{1-\frac{x_k^2}{a^2}-\frac{y_l^2}{b^2}}}dz\right)\Delta y\,\Delta x.$$

Therefore the volume of the solid bounded by the xy plane, the xz-plane, the planes whose equations are $x=x_k$ and $x=x_{k+1}$, and the cylindrical surface one of whose bases is the curve $LKFB_k$

$$=\left\{\underset{m=\infty}{\text{limit}}\sum_{y=0}^{y=b\sqrt{1-\frac{x_k^2}{a^2}}}\left(\int_0^{c\sqrt{1-\frac{x_k^2}{a^2}-\frac{y^2}{b^2}}}dz\right)\Delta y\right\}\Delta x$$

$$=\left\{\int_0^{b\sqrt{1-\frac{x_k^2}{a^2}}}\left(\int_0^{c\sqrt{1-\frac{x_k^2}{a^2}-\frac{y^2}{b^2}}}dz\right)dy\right\}\Delta x.$$

Therefore the required volume

$$=8\left[\underset{n=\infty}{\text{limit}}\sum_{x=0}^{x=a}\left\{\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\left(\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dz\right)dy\right\}\Delta x\right]$$

$$=8\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}dx\,dy\,dz$$

$$=8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\left(\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right)dx\,dy.$$

To integrate with respect to y, let $y = b\sqrt{1-\frac{x^2}{a^2}}\cos\phi$.

$\therefore\ dy = -b\sqrt{1-\frac{x^2}{a^2}}\sin\phi\, d\phi$. Also, $\phi = \frac{\pi}{2}$ when $y=0$, and

$\phi = 0$ when $y = b\sqrt{1-\frac{x^2}{a^2}}$.

$$\therefore\ 8c\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy$$

$$= -8\,bc\int_0^a\int_{\frac{\pi}{2}}^0\left(1-\frac{x^2}{a^2}\right)\sin^2\phi\,dx\,d\phi.$$

Therefore the required volume $= 2\pi bc\int_0^a\left(1-\frac{x^2}{a^2}\right)dx$

$$= \tfrac{4}{3}\pi abc.$$

SURFACES

220. Let $z = f(x, y)$ be the equation of a surface that cuts the xy-plane, the xz-plane, and the planes whose equations are $x = a$ and $x = b$ in curves which together inclose a region on the surface. Suppose that $f(x, y)$ is single valued and continuous for all values of x and y which together determine a point in the region of the xy-plane bounded by the curve of intersection of the surface and xy-plane, and the lines whose equations are $x = a$ and $x = b$. To find the area of the surface bounded by the xy-plane, the xz-plane, and the planes whose equations are $x = a$ and $x = b$.

221. Before proceeding to a consideration of the problem stated in the preceding article, we shall first show that the angle γ which the tangent plane to the surface $z = f(x, y)$ makes with the xy-plane at a point whose coördinates are (x_0, y_0, z_0) is such that $\sec\gamma = \left(\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\right)\Bigg|_{\substack{x=x_0\\ y=y_0}}$.

Let P with coördinates (x_0, y_0, z_0) be a point on the surface. (Fig. 95.) Through P draw a tangent plane to the surface and also planes parallel to the coördinate planes. Let $PM = \Delta x$, and $PN = \Delta y$. Through M draw a plane parallel to the yz-plane, and through N draw a plane parallel to the xz-plane. Let these planes meet the tangent plane at P in lines on which are the points M' and N' respectively, as indicated. Through P draw the line TPS perpendicular to the tangent plane.

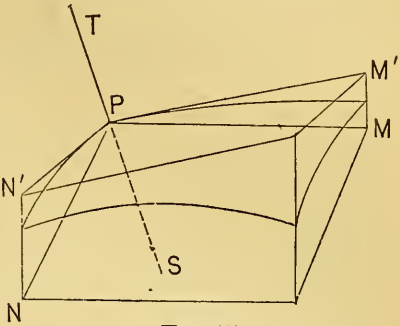


Fig. 95.

Let $MPM' = \phi$, and $N'PN = \psi$. Then the direction cosines of PM' are $\cos\phi$, 0, and $\sin\phi$, and of PN' are 0, $\cos\psi$, and $\sin\psi$. Let the line TPS have the direction angles α, β, and γ. The angle γ, since it is the angle between the lines perpendicular to the tangent plane and the xy-plane respectively, is the angle between the two planes.

Since TPS is normal to the tangent plane it is perpendicular to PM' and PN'.

$$\therefore\ \cos\alpha\cos\phi + \cos\gamma\sin\phi = 0,$$

and
$$\cos\beta\cos\psi + \cos\gamma\sin\psi = 0.$$

Substitute in
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

$$\therefore\ \cos^2\gamma\tan^2\phi + \cos^2\gamma\tan^2\psi + \cos^2\gamma = 1.$$

$$\therefore\ \cos^2\gamma = \frac{1}{1 + \tan^2\phi + \tan^2\psi}.$$

$$\therefore\ \sec^2\gamma = 1 + \tan^2\phi + \tan^2\psi.$$

$$\therefore\ \sec\gamma = \sqrt{1 + \tan^2\phi + \tan^2\psi}.$$

By Art. 127, $\tan\phi = \left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=x_0\\y=y_0}}$, and $\tan\psi = \left.\dfrac{\partial z}{\partial y}\right|_{\substack{x=x_0\\y=y_0}}$.

$$\therefore\ \sec\gamma = \left.\left(\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\right)\right|_{\substack{x=x_0\\y=y_0}}.$$

222. To return to the problem of Art. 220.

Suppose that the surface is as in Fig. 96. Let OA and OB represent a and b respectively. Divide AB into n equal parts. Call each part Δx. At A_k and A_{k+1} draw planes perpendicular to the x-axis to meet the surface in B_kL and $B_{k+1}M$ respectively.

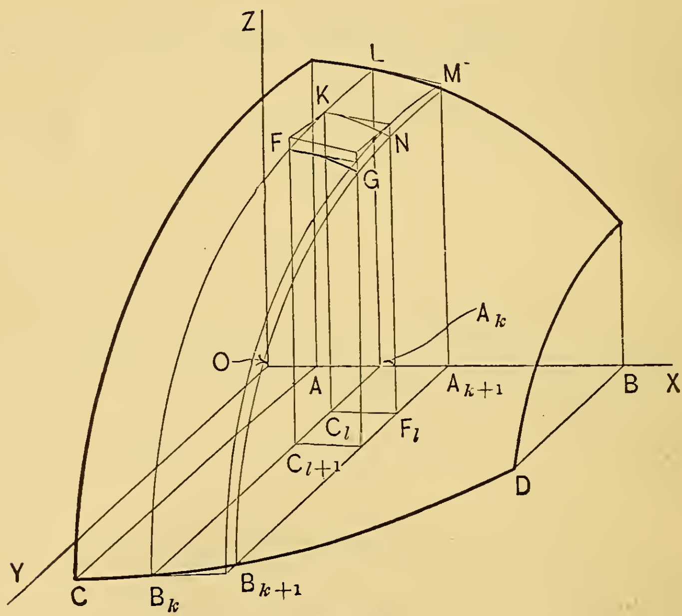


FIG. 96.

Divide A_kB_k into m equal parts. Call each part Δy. At C_l and C_{l+1} draw planes perpendicular to the line A_kB_k to meet the surface in KN and FG respectively. At K draw the tangent plane to the surface. Denote the angle which this tangent plane makes with the xy-plane by γ.

Denote the abscissa OA_k by x_k. Denote the ordinate A_kC_l by y_l.

In $z = f(x, y)$, let $z = 0$ and we have the equation of the curve $CB_kB_{k+1}D$. Solve $f(x, y) = 0$ for y. Suppose that the result is $y = \phi(x)$. Then the coördinates of B_k are $(x_k, \phi(x_k), 0)$.

The area of the tangent plane cut off by the four planes C_lF, C_lN, $C_{l+1}G$, and F_lG is

$$\sec\gamma\, \Delta y\, \Delta x, \text{ or } \left(\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\right)\Bigg|_{\substack{x=x_0\\ y=y_0}} \Delta y\, \Delta x.$$

Let y_l take in succession the values $0, y_1, y_2, \cdots, y_{m-1}$. Then the sum of the areas of all the tangent planes that can be described like the above one on the m equal parts of division of the line A_kB_k

$$= \left[\left(\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\right)\Bigg|_{\substack{x=x_k\\ y=y_0}} + \left(\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\right)_{\substack{x=x_k\\ y=y_1}} + \right.$$

$$\left. \cdots + \left(\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\right)\Bigg|_{\substack{x=x_k\\ y=y_{m-1}}}\right] \Delta y\, \Delta x,$$

$$\text{or } \left[\sum_{y=0}^{y=\phi(x_k)} \left(\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\right)\Bigg|_{x=x_k} \Delta y\right] \Delta x.$$

The limit which this sum approaches as m increases without limit is the area of as much of the curved surface formed by a tangent line to the given surface which moves with its point of contact on the curve $LKFB_k$ as is contained between the planes whose equations are $x = x_k$ and $x = x_{k+1}$. Then the area of the surface $= \left[\lim_{m=\infty} \sum_{y=0}^{y=\phi(x_k)} \left(\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\right)\Bigg|_{x=x_k} \Delta y\right] \Delta x$, or $\left(\int_0^{\phi(x_k)} \left(\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\right)\Bigg|_{x=x_k} dy\right) \Delta x$. Let x_k take in succession the values $a, x_1, x_2, \cdots, x_{n-1}$. Then the sum of the areas of all the surfaces described above that can be described on the n equal parts of division of the line AB

$$=\left[\int_0^{\phi(a)}\left(\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\right)\Bigg|_{x=a} dy\right.$$

$$+\int_0^{\phi(x_1)}\left(\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\right)\Bigg|_{x=x_1} dy$$

$$\left.+\cdots+\int_0^{\phi(x_{n-1})}\left(\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\right)\Bigg|_{x=x_{n-1}} dy\right]\Delta x$$

$$=\sum_{x=a}^{x=b}\left(\int_0^{\phi(x)}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dy\right)\Delta x.$$

It can be shown by an investigation which would be beyond the scope of this book that the limit which this sum approaches as n increases without limit is the area of the given surface. Assuming that this limit is the area of the given surface, we have that the area of the given surface

$$=\lim_{n=\infty}\left[\sum_{x=a}^{x=b}\left(\int_0^{\phi(x)}\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dy\right)\right]\Delta x$$

$$=\int_a^b\int_0^{\phi(x)}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy.$$

223. EXAMPLE. Find the area of the surface of the sphere

$$x^2+y^2+z^2=a^2.$$

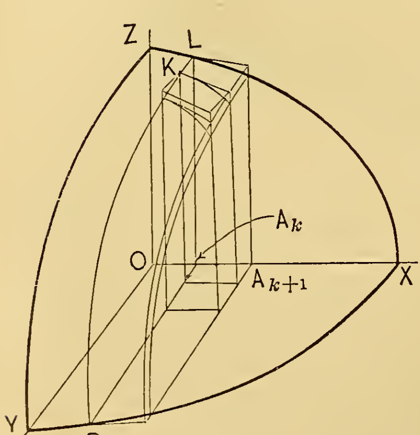


FIG. 97.

Consider the part of the sphere in the first octant.

Make the same construction as in the general case.

Since $z=\sqrt{a^2-x^2-y^2}$,

$$\therefore\ \frac{\partial z}{\partial x}=\frac{-x}{\sqrt{a^2-x^2-y^2}},$$

and $$\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{a^2-x^2-y^2}}.$$

$$\therefore \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}=\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}$$

$$=\frac{a}{\sqrt{a^2-x^2-y^2}}.$$

The area of as much of the curved surface formed by a tangent line to the given surface which moves with its point of contact on the curve LKB_k as is contained between the planes whose equations are $x=x_k$ and $x=x_{k+1}$

$$=\left[\underset{m=\infty}{\text{limit}}\sum_{y=0}^{y=\sqrt{a^2-x_k{}^2}}\frac{a\,\Delta y}{\sqrt{a^2-x_k{}^2-y^2}}\right]\Delta x$$

$$=\left(\int_0^{\sqrt{a^2-x_k{}^2}}\frac{a\,dy}{\sqrt{a^2-x_k{}^2-y^2}}\right)\Delta x.$$

Therefore the required area

$$=8\left[\underset{n=\infty}{\text{limit}}\sum_{x=0}^{x=a}\left(\int_0^{\sqrt{a^2-x^2}}\frac{a\,dy}{\sqrt{a^2-x^2-y^2}}\right)\Delta x\right]$$

$$=8\,a\int_0^a\int_0^{\sqrt{a^2-x^2}}\frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$$

$$=4\,\pi\,a^2.$$

EXERCISES

1. Find the volume inclosed by the xy-plane, the plane $z=mx$, and the cylinder $x^2+y^2=a^2$. *Ans.* $\frac{2}{3}\,a^3m$.

2. Find the volume common to the two cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$. *Ans.* $\frac{16}{3}\,a^3$.

3. Find the volume common to the paraboloid of revolution $y^2+z^2=4\,ax$ and the cylinder $x^2+y^2=2\,ax$.

Ans. $\left[2\,\pi+\frac{16}{3}\right]a^3$.

4. Find the volume inclosed by the surface $\left(\frac{x}{a}\right)^{\frac{1}{2}}+\left(\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{z}{c}\right)^{\frac{1}{2}}=1$ and the positive sides of the three coördinate planes.

Ans. $\frac{abc}{90}$.

5. Find the area of the surface cut off from the sphere $x^2+y^2+z^2=a^2$ by the cylinder $x^2+y^2=ax$. *Ans.* $2[\pi-2]\,a^2$.

6. Find the area of the surface cut off from the cylinder $x^2+y^2=ax$ by the sphere $x^2+y^2+z^2=a^2$. *Ans.* $4\,a^2$.

7. Find the area of the surface inclosed by the two cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$. *Ans.* $16\,a^2$.

8. Find the area of the convex surface described in Exercise 1. *Ans.* $4\,a^2m$.

9. The equation of the base of a cylinder is $y^{\frac{2}{3}}+z^{\frac{2}{3}}=a^{\frac{2}{3}}$ and its axis is parallel to the x-axis. Find the area of the portion of the surface bounded by a curve whose projection on the xy-plane is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$. *Ans.* $\frac{12}{5}\,a^2$.

10. A square hole is cut from a sphere, the axis of the hole coinciding with a diameter of the sphere. The radius of the sphere is a and a diagonal of the hole is $2\sqrt{2}\,b$. Find the area of the surface cut from the sphere by the hole.

Ans. $8\,a\left[2\,b\sin^{-1}\frac{b}{\sqrt{a^2-b^2}}-a\sin^{-1}\frac{b^2}{a^2-b^2}\right]$.

CHAPTER XXIX

SOME METHODS OF APPROXIMATE INTEGRATION

224. In the problems thus far considered, involving the definite integral of an expression, the definite integral could always be expressed as the difference between two values of an indefinite integral. In many important cases, however, the definite integral cannot be so expressed. When such a case arises, we are usually content with an approximate value of the definite integral, found by developing the integrand into a power series by Taylor's Theorem or one of its special cases, Maclauren's or the Binomial Theorem, and integrating the first few terms of the series between the given limits. Two cases where approximate values can be found in this manner will now be investigated. From their importance they are given special names,—the Elliptic Integrals of the First and Second Classes respectively.

Case I. It will be shown later that in a pendulum of length a, if the bob starts from the lowest point of its path with the velocity which it would acquire if it fell freely in a vacuum through a distance h, the time it takes to reach a point whose ordinate is y is

$$t=\frac{a}{\sqrt{2g}}\int_0^y\frac{dy}{\sqrt{(h-y)(2ay-y^2)}}, \tag{1}$$

where the equation of the path is $x^2+y^2-2ay=0$.

If $h<2a$, let $x^2=\frac{y}{h}$, and (1) becomes

$$t=\sqrt{\frac{a}{g}}\int_0^x\frac{dx}{\sqrt{\left(1-x^2\right)\left(1-\frac{h}{2a}x^2\right)}},$$

where the integral is of the form

$$\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

k^2 being positive and less than 1.

If $h > 2a$, let $x^2 = \frac{y}{2a}$, and (1) becomes

$$t = a\sqrt{\frac{2}{gh}}\int_0^x \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-\frac{2a}{h}x^2\right)}},$$

where the integral is of the form

$$\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

k^2 being positive and less than 1.

Case II. It can be readily shown by the methods of Chapter XXII, that the length of the arc of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ contained between the point (o, b) and the point (x, y) on the curve is

$$s = \int_0^x \sqrt{\frac{a^2-e^2x^2}{a^2-x^2}}\,dx, \tag{1}$$

where e is the eccentricity of the ellipse.

Replace $\frac{x}{a}$ by x, and (1) becomes

$$s = a\int_0^x \sqrt{\frac{1-e^2x^2}{1-x^2}}\,dx,$$

where the integral is of the form

$$\int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx,$$

k^2 being positive and less than 1.

225. Definitions. The expression $\int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ is called an Elliptic Integral of the First Class. It is denoted by $F(k, x)$.

The expression $\int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$ is called an Elliptic Integral of the Second Class. It is denoted by $E(k, x)$.

The expressions

$$F(k, 1) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \text{ and } E(k, 1) = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx$$

are called the Complete Elliptic Integrals of the First and Second Classes respectively. They are usually denoted by K and E respectively.

226. The substitution of $x = \sin\phi$ in the Elliptic Integrals reduces them to the forms

$$F(k, \phi) = \int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}, \text{ and } E(k, \phi) = \int_0^\phi \sqrt{1-k^2\sin^2\phi}\,d\phi$$

respectively.

227. An approximate value of $F(k, \phi)$.

Develop $(1-k^2\sin^2\phi)^{-\frac{1}{2}}$ into an infinite series by the Binomial Theorem.

Since k is less than 1, $k^2\sin^2\phi$ is less than 1 for all values of ϕ.

$$\therefore\ (1-k^2\sin^2\phi)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}k^2\sin^2\phi + \frac{1\cdot 3}{2^2\lfloor 2}k^4\sin^4\phi + \cdots \qquad (1)$$

for all values of ϕ. (See Art. 95.)

If the right-hand member of (1) be integrated term by term with respect to ϕ between the limits 0 and ϕ, the resulting series is

$$\phi + \frac{k^2}{4}(\phi - \sin\phi\cos\phi) - \tfrac{3}{32}k^4\sin^3\phi\cos\phi$$
$$+ \tfrac{9}{64}k^4(\phi - \sin\phi\cos\phi)\cdots. \qquad (2)$$

Series (2) gives an approximate value of $F(k, \phi)$ for given values of k and ϕ, the degree of accuracy depending on the number of terms computed in the series.

228. Example. Find an approximate value of $F\left(\frac{\sqrt{2}}{2}, \frac{\pi}{6}\right)$.

For $k = \frac{\sqrt{2}}{2}$, $\phi = \frac{\pi}{6}$, series (2) becomes

$$\frac{\pi}{6} + \frac{1}{8}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) - \frac{3}{32} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{\sqrt{3}}{2} + \frac{9}{64} \cdot \frac{1}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) \cdots$$

$$= 0.52359 \cdots + 0.01132 \cdots - 0.00253 \cdots + 0.00318 \cdots$$

$$= 0.53556 \cdots.$$

An approximate value of $F\left(\frac{\sqrt{2}}{2}, \frac{\pi}{6}\right)$ is therefore 0.53556.

Note. The approximate values of this chapter are not approximate in the sense that they are correct to the number of decimal places given. They are merely results found in the attempt to get the actual values of the integrals.

229. An approximate value of $E(k, \phi)$.

Develop $(1 - k^2 \sin^2 \phi)^{\frac{1}{2}}$ into an infinite series by the Binomial Theorem.

$$\therefore\ (1 - k^2 \sin^2 \phi)^{\frac{1}{2}} = 1 - \tfrac{1}{2} k^2 \sin^2 \phi - \frac{1 \cdot 1}{2^2 \underline{|2}} k^4 \sin^4 \phi + \cdots \qquad (1)$$

for all values of ϕ.

Integrate each term of the right-hand member of (1) with respect to ϕ between the limits 0 and ϕ. The resulting series is

$$\phi - \frac{k^2}{4}(\phi - \sin \phi \cos \phi) + \frac{k^4}{32} \sin^3 \phi \cos \phi$$

$$- \tfrac{3}{64} k^4 (\phi - \sin \phi \cos \phi) + \cdots. \qquad (2)$$

Series (2) gives an approximate value of $E(k, \phi)$ for given values of k and ϕ, the degree of accuracy depending, as in the case of $F(k, \phi)$, on the number of terms computed in the series.

230. EXAMPLE. Find an approximate value of $E\left(\frac{\sqrt{2}}{2}, \frac{\pi}{6}\right)$.

For $k = \frac{\sqrt{2}}{2}$, $\phi = \frac{\pi}{6}$, series (2) becomes

$$\frac{\pi}{6} - \frac{1}{8}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + \frac{1}{32} \cdot \frac{1}{4} \cdot \frac{1}{8} \cdot \frac{\sqrt{3}}{2} - \frac{3}{64} \cdot \frac{1}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) \cdots$$

$$= 0.52359 \cdots - 0.01132 \cdots + 0.00084 \cdots - 0.00106 \cdots$$

$$= 0.51205 \cdots.$$

An approximate value of $E\left(\frac{\sqrt{2}}{2}, \frac{\pi}{6}\right)$ is therefore 0.51205.

231. Find the length of the arc of the lemniscate $r^2 = a^2 \cos 2\theta$ contained between the point $(a, 0)$, and the point (r, θ) on the curve.

Since $r^2 = a^2 \cos 2\theta$, $\therefore r = \pm a\sqrt{\cos 2\theta}$. Take the $+$ sign for r. $\therefore \frac{dr}{d\theta} = \frac{a \sin 2\theta}{\sqrt{\cos 2\theta}}$.

$$\therefore \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2 \cos 2\theta + \frac{a^2 \sin^2 2\theta}{\cos 2\theta}} = \frac{a}{\sqrt{\cos 2\theta}}.$$

$$\therefore s = a\int_0^\theta \frac{d\theta}{\sqrt{\cos 2\theta}}$$

$$= a\int_0^\theta \frac{d\theta}{\sqrt{1 - 2\sin^2\theta}}.$$

This is not an Elliptic Integral, since k^2, or 2, is not less than 1. It can, however, be put in such a form by means of the transformation $2\sin^2\theta = \sin^2\phi$.

Let $\qquad 2\sin^2\theta = \sin^2\phi. \quad \therefore \sin\theta = \frac{\sqrt{2}}{2}\sin\phi.$

$$\therefore s = a\frac{\sqrt{2}}{2}\int_0^\phi \frac{d\phi}{\sqrt{1 - \frac{1}{2}\sin^2\phi}}$$

$$= a\frac{\sqrt{2}}{2} F\left(\frac{\sqrt{2}}{2}, \phi\right).$$

For given values of ϕ, s can be found approximately in terms of a by the method of Art. 227.

232. Find the length of the arc of the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ contained between the point $(a, 0)$, and the point (x, y) on the curve.

Since $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, $\therefore\ x=\pm\dfrac{a}{b}\sqrt{y^2+b^2}$. Take the $+$ sign for x. $\therefore\ \dfrac{dx}{dy}=\dfrac{a}{b}\dfrac{y}{\sqrt{y^2+b^2}}$.

$$\therefore\ \sqrt{1+\left(\frac{dx}{dy}\right)^2}=\sqrt{1+\frac{a^2y^2}{b^2(y^2+b^2)}}=\frac{\sqrt{(a^2+b^2)y^2+b^4}}{\sqrt{b^2y^2+b^4}}$$

$$=\left[\frac{1+\dfrac{(a^2+b^2)y^2}{b^4}}{1+\dfrac{y^2}{b^2}}\right]^{\frac{1}{2}}.$$

$$\therefore\ s=\int_0^y\left\{\frac{1+\dfrac{a^2e^2y^2}{b^4}}{1+\dfrac{y^2}{b^2}}\right\}^{\frac{1}{2}}dy, \qquad (1)$$

where e is the eccentricity of the hyperbola.

In (1), let $\dfrac{aey}{b^2}=\tan\phi$.

$$\therefore\ s=\frac{b^2}{ae}\int_0^\phi\frac{\sec^2\phi\,d\phi}{\sqrt{1-\dfrac{1}{e^2}\sin^2\phi}}. \qquad (2)$$

Since $\sec^2\phi=1+\tan^2\phi$,

$$\therefore\ s=\frac{b^2}{ae}\int_0^\phi\frac{d\phi}{\sqrt{1-\dfrac{1}{e^2}\sin^2\phi}}+\frac{b^2}{ae}\int_0^\phi\frac{\tan^2\phi\,d\phi}{\sqrt{1-\dfrac{1}{e^2}\sin^2\phi}}$$

$$=\frac{b^2}{ae}F\left(\frac{1}{e},\phi\right)+\frac{b^2}{ae}\int_0^\phi\frac{\tan^2\phi\,d\phi}{\sqrt{1-\dfrac{1}{e^2}\sin^2\phi}}.$$

$$\text{Now } \frac{b^2}{ae}\int_0^\phi \frac{\tan^2\phi\, d\phi}{\sqrt{1-\frac{1}{e^2}\sin^2\phi}}$$

$$= \frac{b^2}{ae}\int_0^\phi \frac{\left(-\frac{a^2e^2}{b^2}+\frac{a^2}{b^2}\sin^2\phi+\frac{a^2e^2}{b^2}-\frac{a^2}{b^2}\sin^2\phi+\tan^2\phi\right)d\phi}{\sqrt{1-\frac{1}{e^2}\sin^2\phi}}$$

$$= -ae\int_0^\phi \frac{\left(1-\frac{1}{e^2}\sin^2\phi\right)d\phi}{\sqrt{1-\frac{1}{e^2}\sin^2\phi}}$$

$$+ae\int_0^\phi \frac{\left(1-\frac{1}{e^2}\sin^2\phi+\frac{b^2}{a^2e^2}\tan^2\phi\right)d\phi}{\sqrt{1-\frac{1}{e^2}\sin^2\phi}}$$

$$= -aeE\left(\frac{1}{e},\phi\right)+ae\int_0^\phi \frac{\left(1-\frac{1}{e^2}\sin^2\phi+\frac{e^2-1}{e^2}\tan^2\phi\right)d\phi}{\sqrt{1-\frac{1}{e^2}\sin^2\phi}}$$

$$= -aeE\left(\frac{1}{e},\phi\right)+ae\tan\phi\sqrt{1-\frac{1}{e^2}\sin^2\phi},$$

since the last integral becomes $\int_0^u du$ on the substitution of $u=\tan\phi\sqrt{1-\frac{1}{e^2}\sin^2\phi}$.

$$\therefore\ s=\frac{b^2}{ae}F\left(\frac{1}{e},\phi\right)-aeE\left(\frac{1}{e},\phi\right)+ae\tan\phi\sqrt{1-\frac{1}{e^2}\sin^2\phi}.$$

233. Tables giving the values of the Elliptic Integrals of the First and Second Classes to ten places of decimals are given in Legendre's *Traité des Fonctions Elliptiques.* In the next article is given a small three-place table compiled from these which may be used in solving the following exercises.

EXERCISES

1. Compare the results found for $F\left(\frac{\sqrt{2}}{2}, \frac{\pi}{6}\right)$ and $E\left(\frac{\sqrt{2}}{2}, \frac{\pi}{6}\right)$ in Arts. 228 and 230 with those found by using the tables.

2. Find the lengths of the arcs of the ellipses $\frac{x^2}{16}+\frac{y^2}{25}=1$ and $\frac{x^2}{36}+\frac{y^2}{16}=1$. *Ans.* 28.32; 31.68.

3. Find the length of the arc of the lemniscate given in Art. 231. *Ans.* $5.243\,a$.

4. Find the length of the arc of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ contained between the points (4, 0) and $(8, 3\sqrt{3})$.

Ans. 6.725.

Art. 234. $F(k, \phi)$

ϕ	$k=0$ sin 0°	$k=0.1$ sin 6°	$k=0.2$ sin 12°	$k=0.3$ sin 18°	$k=0.4$ sin 24°	$k=0.5$ sin 30°	$k=0.6$ sin 37°	$k=0.7$ sin 45°	$k=0.8$ sin 53°	$k=0.9$ sin 64°	$k=1$ sin 90°
0°	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5°	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087
10°	0.175	0.175	0.175	0.175	0.175	0.175	0.175	0.175	0.175	0.175	0.175
15°	0.262	0.262	0.262	0.262	0.262	0.263	0.263	0.263	0.264	0.264	0.265
20°	0.349	0.349	0.349	0.350	0.350	0.351	0.352	0.353	0.354	0.355	0.356
25°	0.436	0.436	0.437	0.438	0.439	0.440	0.441	0.443	0.445	0.448	0.451
30°	0.524	0.524	0.525	0.526	0.527	0.529	0.532	0.536	0.539	0.544	0.549
35°	0.611	0.611	0.612	0.614	0.617	0.622	0.624	0.630	0.636	0.644	0.653
40°	0.698	0.699	0.700	0.703	0.707	0.712	0.718	0.727	0.736	0.748	0.763
45°	0.785	0.786	0.789	0.792	0.798	0.804	0.814	0.826	0.839	0.858	0.881
50°	0.873	0.874	0.877	0.882	0.889	0.898	0.911	0.928	0.947	0.974	1.011
55°	0.960	0.961	0.965	0.972	0.981	0.993	1.010	1.034	1.060	1.099	1.154
60°	1.047	1.049	1.054	1.062	1.074	1.090	1.112	1.142	1.178	1.233	1.317
65°	1.134	1.137	1.143	1.153	1.168	1.187	1.215	1.254	1.302	1.377	1.506
70°	1.222	1.224	1.232	1.244	1.262	1.285	1.320	1.370	1.431	1.534	1.735
75°	1.309	1.312	1.321	1.336	1.357	1.385	1.426	1.488	1.566	1.703	2.028
80°	1.396	1.400	1.410	1.427	1.452	1.485	1.534	1.608	1.705	1.885	2.436
85°	1.484	1.487	1.499	1.519	1.547	1.585	1.643	1.731	1.848	2.077	3.131
90°	1.571	1.575	1.588	1.610	1.643	1.686	1.752	1.854	1.993	2.275	∞

$E(k, \phi)$

ϕ	$k=0$ sin 0°	$k=0.1$ sin 6°	$k=0.2$ sin 12°	$k=0.3$ sin 18°	$k=0.4$ sin 24°	$k=0.5$ sin 30°	$k=0.6$ sin 37°	$k=0.7$ sin 45°	$k=0.8$ sin 53°	$k=0.9$ sin 64°	$k=1$ sin 90°
0°	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5°	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087	0.087
10°	0.175	0.175	0.174	0.174	0.174	0.174	0.174	0.174	0.174	0.174	0.174
15°	0.262	0.262	0.262	0.262	0.261	0.261	0.261	0.260	0.260	0.259	0.259
20°	0.349	0.349	0.349	0.348	0.348	0.347	0.347	0.346	0.345	0.343	0.342
25°	0.436	0.436	0.436	0.435	0.434	0.433	0.431	0.430	0.428	0.425	0.423
30°	0.524	0.523	0.523	0.521	0.520	0.518	0.515	0.512	0.509	0.505	0.500
35°	0.611	0.610	0.609	0.607	0.605	0.602	0.598	0.593	0.588	0.581	0.574
40°	0.698	0.698	0.696	0.693	0.690	0.685	0.679	0.672	0.664	0.654	0.643
45°	0.785	0.785	0.782	0.779	0.773	0.767	0.759	0.748	0.737	0.723	0.707
50°	0.873	0.872	0.869	0.864	0.857	0.848	0.837	0.823	0.808	0.789	0.766
55°	0.960	0.959	0.955	0.948	0.939	0.928	0.914	0.895	0.875	0.850	0.819
60°	1.047	1.046	1.041	1.032	1.021	1.008	0.989	0.965	0.940	0.907	0.866
65°	1.134	1.132	1.126	1.116	1.103	1.086	1.063	1.033	1.001	0.960	0.906
70°	1.222	1.219	1.212	1.200	1.184	1.163	1.135	1.099	1.060	1.008	0.940
75°	1.309	1.306	1.297	1.283	1.264	1.240	1.207	1.163	1.117	1.053	0.966
80°	1.396	1.393	1.383	1.367	1.344	1.316	1.277	1.227	1.172	1.095	0.985
85°	1.484	1.480	1.468	1.450	1.424	1.392	1.347	1.289	1.225	1.135	0.996
90°	1.571	1.566	1.554	1.533	1.504	1.467	1.417	1.351	1.278	1.173	1.000

From Art. 190 we see that a definite integral, $\int_a^b F(x)\,dx$, can be interpreted as the area inclosed by the curve $y = F(x)$, the x-axis, and the ordinates corresponding to the abscissas a and b respectively. The value of a definite integral, $\int_a^b F(x)\,dx$, can therefore be found approximately by either one of the following trapezoidal methods, or by Simpson's Rule.

TRAPEZOIDAL METHODS

235. Plot the curve $y = F(x)$. Suppose that it is as in Fig. 98. Let OA and OB represent the abscissas a and b respectively. Divide AB into $2n$ equal parts. Call each part Δx. At A, B, and $A_1, A_2, A_3, \cdots, A_{2n-1}$, the points of division

of AB, erect ordinates to meet the curve in C, D, and B_1, B_2, B_3, $\cdots$, B_{2n-1} respectively. Join CB_1, B_1B_2, B_2B_3, $\cdots$, $B_{2n-1}D$. Denote the ordinates of the points C, B_1, B_2, B_3, $\cdots$, D by y_1, y_2, y_3, $\cdots$, y_{2n+1} respectively.

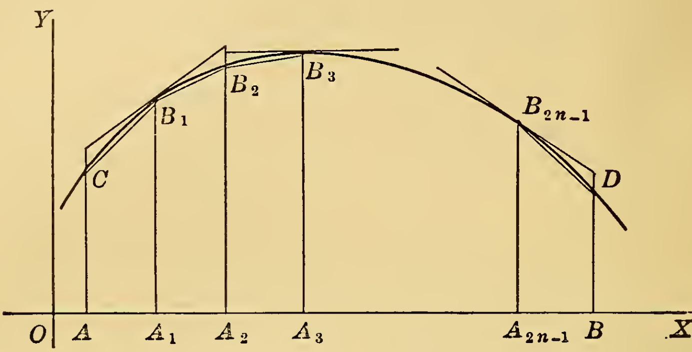


FIG. 98.

By geometry, the area of a trapezoid is the product of one half the sum of the bases by the height. Therefore the sum of the areas of the trapezoids ACB_1A_1, $A_1B_1B_2A_2$, $A_2B_2B_3A_3$, $\cdots$, $A_{2n-1}B_{2n-1}DB$

$$= \tfrac{1}{2}[(y_1 + y_2) + (y_2 + y_3) + (y_3 + y_4) + \cdots + (y_{2n} + y_{2n+1})]\Delta x$$

$$= \Delta x[\tfrac{1}{2}y_1 + y_2 + y_3 + y_4 + \cdots + y_{2n} + \tfrac{1}{2}y_{2n+1}].$$

Therefore an approximate value of $\int_a^b F(x)\,dx$ is

$$\Delta x[\tfrac{1}{2}y_1 + y_2 + y_3 + y_4 + \cdots + y_{2n} + \tfrac{1}{2}y_{2n+1}].$$

In the above, the number of equal divisions of AB was taken even. It could however as well be taken odd.

In Fig. 98, the curve was assumed to be concave downwards. In such a case the approximation is obviously less than the value of the integral. If the curve is concave upwards, the approximation is greater than the value of the integral.

Obviously, the greater the number of parts into which AB is divided the closer will be the approximation.

A second trapezoidal formula may be derived as follows:

At $B_1, B_3, \cdots, B_{2n-1}$ (Fig. 98), draw tangents to the curve to meet the ordinates produced.

The sum of the areas of the trapezoids formed by the ordinates or ordinates produced, the x-axis, and the tangent lines thus drawn $= 2\,\Delta x[y_2 + y_4 + y_6 + \cdots + y_{2n}]$.

Therefore an approximate value of $\int_a^b F(x)\,dx$

$$= 2\,\Delta x[y_2 + y_4 + y_6 + \cdots + y_{2n}].$$

In this case it is necessary that the number of divisions be even.

In this case, if the curve is concave downwards, the approximation is greater than the integral, and if concave upwards, less.

As before, the greater the number of parts into which AB is divided the closer will be the approximation.

Usually the mean of (1) and (2) will give a result closer than either.

236. In illustration of the above methods, consider the following example.

Find the approximate values of $\int_1^5 \frac{dx}{x}$ if $2n = 12$.

Here $\Delta x = \frac{5-1}{12} = \frac{1}{3}$; $y_1 = \frac{1}{1} = 1$; $y_2 = \frac{1}{1\frac{1}{3}} = \frac{3}{4}$; $y_3 = \frac{1}{1\frac{2}{3}} = \frac{3}{5}$; $y_4 = \frac{1}{2}$; $y_5 = \frac{1}{2\frac{1}{3}} = \frac{3}{7}$; $y_6 = \frac{1}{2\frac{2}{3}} = \frac{3}{8}$; $y_7 = \frac{1}{3}$; $y_8 = \frac{1}{3\frac{1}{3}} = \frac{3}{10}$; $y_9 = \frac{1}{3\frac{2}{3}} = \frac{3}{11}$; $y_{10} = \frac{1}{4}$; $y_{11} = \frac{1}{4\frac{1}{3}} = \frac{3}{13}$; $y_{12} = \frac{1}{4\frac{2}{3}} = \frac{3}{14}$; $y_{13} = \frac{1}{5}$.

Therefore an approximate value of $\int_1^5 \frac{dx}{x}$,

by the first method, $= \frac{1}{3}[\frac{1}{2} + \frac{3}{4} + \frac{3}{5} + \frac{1}{2} + \frac{3}{7} + \frac{3}{8} + \frac{1}{3} + \frac{3}{10} + \frac{3}{11} + \frac{1}{4} + \frac{3}{13} + \frac{3}{14} + \frac{1}{10}]$

$$= 1.6182,$$

and by the second method,

$$= \tfrac{2}{3}[\tfrac{3}{4} + \tfrac{1}{2} + \tfrac{3}{8} + \tfrac{3}{10} + \tfrac{1}{4} + \tfrac{3}{14}]$$
$$= 1.5929.$$

The mean of these values $= 1.6055$.

By integration, $$\int_1^5 \frac{dx}{x} = \log_e 5 = 1.6094.$$

The error by the first method is therefore less than 0.009, by the second method, than 0.007, and by the taking the mean of the two results, than 0.004.

SIMPSON'S METHOD OR RULE

237. This method is based on the following theorem:

If three points P, Q, and R on the parabola whose equation is in the form $y = ax^2 + bx + c$ be such that the ordinates of P and R are equidistant from Q, then the area inclosed by this parabola, the x-axis, and the ordinates of P and R is

$$\frac{h}{3}[y' + 4y'' + y'''], \tag{1}$$

where y', y'', y''' are the ordinates of P, Q, and R respectively.

Proof. Denote the abscissas of P, Q, and R by $x' - h$, x' and $x' + h$ respectively.

The required area $= \int_{x'-h}^{x'+h} (ax^2 + bx + c)\,dx.$

$$= h\left[\frac{a}{3}(6x'^2 + 2h^2) + 2bx' + 2c\right]. \tag{2}$$

Since P, Q, and R lie on the parabola,

$$y' = a(x' - h)^2 + b(x' - h) + c,$$
$$y'' = ax'^2 + bx' + c,$$
$$y''' = a(x' + h)^2 + b(x' + h) + c,$$

Solve these equations for a, b, and c.

$$\therefore a = \frac{1}{2h^2}(y' - 2y'' + y'''),$$

$$b = \frac{1}{2h}(y''' - y') - \frac{x'}{h^2}(y' - 2y'' + y'''),$$

$$c = y'' - \frac{x'}{2h}(y''' - y') + \frac{x'^2}{2h^2}(y' - 2y'' + y''').$$

Substitute these values for a, b, and c in (2) and reduce.

Therefore the required area $= \frac{h}{3}[y' + 4y'' + y''']$, which is the value given in (1). The theorem is therefore proved.

238. Simpson's Rule can now be established as follows:

Pass a parabola through the points C, B_1, and B_2 (Fig. 99), with its transverse axis parallel to the y-axis. The equation

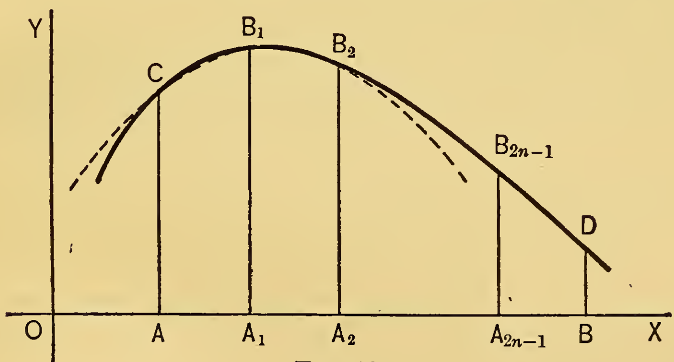


FIG. 99.

will be in the form $y = ax^2 + bx + c$, where a, b, and c have certain numerical values, $a \neq 0$. Therefore, by the above theorem, the area inclosed by this parabola, the x-axis, and the ordinates AC and A_2B_2 is

$$\frac{\Delta x}{3}\left[y_1 + 4y_2 + y_3\right].$$

In like manner, by passing parabolas through $B_2B_3B_4$, $B_4B_5B_6$, and so on up till we reach $B_{2n-2}B_{2n-1}D$, we get the set of areas

$$\frac{\Delta x}{3}\left[y_3 + 4\,y_4 + y_5\right],$$

$$\frac{\Delta x}{3}\left[y_5 + 4\,y_6 + y_7\right],$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{\Delta x}{3}\left[y_{2n-1} + 4\,y_{2n} + y_{2n+1}\right].$$

Therefore, by addition, the sum of the areas of the parabolas

$$= \frac{\Delta x}{3}\Big[y_1 + 2\,(y_3 + y_5 + y_7 + \cdots + y_{2n-1}) + 4\,(y_2 + y_4 + y_6 + \cdots y_{2n}) + y_{2n+1}\Big].$$

Therefore an approximate value of $\int_a^b F(x)\,dx$

$$= \frac{\Delta x}{3}\Big[y_1 + 2\,(y_3 + y_5 + y_7 + \cdots + y_{2n-1}) + 4\,(y_2 + y_4 + y_6 + \cdots + y_{2n}) + y_{2n+1}\Big].$$

Simpson's Rule is slightly more difficult of application than either of the trapezoidal rules, but gives a closer result than either. It will usually give a closer result than that found by taking the mean of the results found by the trapezoidal rules.

When Simpson's Rule is applied, the number of divisions of the line AB must be even.

PLANIMETERS

239. A planimeter is a mechanical contrivance whereby a plane area is measured by passing a tracer around the bounding curves. One in most common use is Amsler's Polar Planimeter, a description of which is given in Carr's *Synopsis of Pure Mathematics*. There is also a discussion on planimeters in Carpenter's *Text-book of Experimental Engineering*.

EXERCISES

1. Calculate an approximate value of $\int_1^5 \frac{dx}{x}$, $2n = 12$, by Simpson's Rule. *Ans.* 1.6098.

For each of the following integrals, for the value of $2n$ set opposite the integral, calculate approximate values by all rules, and compare results with the exact values of the integrals.

2. $\int_1^{10} \log_{10} x\, dx$, $2n = 10$.

Ans. First Trap. Rule, 6.0656; Second Trap. Rule, 6.1374; Mean, 6.1015; Simpson's Rule, 6.0896.

3. $\int_0^2 \frac{dx}{1+x^3}$, $2n = 6$.

Ans. First Trap. Rule, 1.0885; Second Trap. Rule, 1.0946; Mean, 1.0915; Simpson's Rule, 1.0906.

4. $\int_2^6 x^3\, dx$, $2n = 12$.

Ans. First Trap. Rule, 320.89; Second Trap. Rule, 318.22; Mean, 319.56; Simpson's Rule, 320.

CHAPTER XXX

ELEMENTS OF KINEMATICS

For the purpose of completeness, those definitions and fundamental principles of kinematics needed in the subsequent chapters are here briefly treated.

240. Definition. The change of position of a particle from a point P_1 to a point P_2 is called the **displacement** of the particle from P_1 to P_2.

241. A displacement from a point P_1 to a point P_2 is known when, and only when, the magnitude and direction of the straight line joining P_1 and P_2 can be determined completely. A displacement has therefore both length and direction.

The direction on the line in which displacement takes place is indicated by an arrowhead.

242. Let A and B indicate two displacements of a particle, and let P_1 be the initial position of the particle. Draw P_1P_2 equal to A and in the same direction with it. From P_2 draw P_2P_3 equal to B and in the same direction with it.

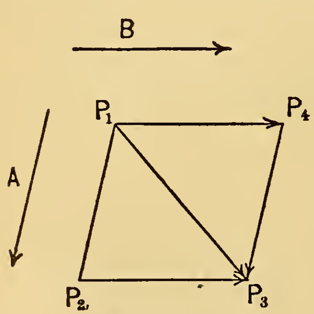


FIG. 100.

The two displacements P_1P_2 and P_2P_3 are equivalent to the single displacement P_1P_3.

Complete the parallelogram $P_1P_2P_3P_4$. Since $P_1P_2 = P_4P_3$, and $P_2P_3 = P_1P_4$, the two displacements P_1P_4 and P_4P_3, or B and A, are equivalent to the single displacement P_1P_3. In the case of two displacements, therefore, the order in which the displacements are made is immaterial.

To find the single displacement which is equivalent to two given displacements, we lay off the two given displacements as two sides of a triangle with the arrowheads the same way round. The remaining side of the triangle with the arrowhead reversed is the single displacement equivalent to the two given displacements.

243. Let A, B, and C be three displacements, and P_1 the initial position of the particle. From P_1 draw P_1P_2 equal to A and in the same direction with it. From P_2 draw P_2P_3 equal to B and in the same direction with it. From P_3 draw P_3P_4 equal to C and in the same direction with it.

The displacements P_1P_2 and P_2P_3 are equivalent to the displacement P_1P_3, and the displacements P_1P_3 and P_3P_4 to the displacement P_1P_4. Then the displacements P_1P_2, P_2P_3, and P_3P_4, or A, B, and C, are equivalent to the single displacement P_1P_4.

Also, as in two displacements, it can be readily shown that the order in which the displacements are made is immaterial.

To find the single displacement which is equivalent to the three given displacements, we lay off the three given displacements as the sides of a quadrilateral with the arrowheads the same way round. The remaining side of the quadrilateral with the arrowhead reversed is the single displacement equivalent to the three given displacements.

244. A displacement can be resolved into any number of component displacements parallel to given lines.

Let P_1P_2 be a given displacement. Let us, for definiteness, resolve it into two displacements parallel to the lines A and B.

From P_1 draw a line parallel to A, and from P_2 draw a line parallel to B. Let P_3 be the point of intersection of these lines.

Then P_1P_2 is resolved into two displacements parallel to A and B.

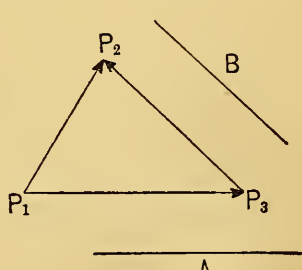


FIG. 101.

245. Definition. The mean velocity of a moving particle during a given time is a quantity which has for direction the direction of the displacement of the particle in that time, and for magnitude the magnitude of the displacement divided by the time.

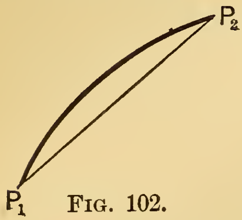


FIG. 102.

Thus, if a particle move on any path from P_1 to P_2 in a given time t, its mean velocity has the direction of the straight line P_1P_2, and the magnitude the length of the line divided by t.

Mean velocity must be distinguished from mean speed. The mean speed of the particle in the above illustration is

$$\frac{\text{arc } P_1P_2}{t}.$$

The mean velocity of a particle in general varies with the time. If it does not, both the direction of the particle and the quotient of the magnitude of its displacement by the time must be constant. In such a case the particle moves in a straight line and with uniform speed.

246. Definitions. A particle which moves in a straight line with uniform speed is said to have **uniform velocity.**

The velocity of a particle at a given instant is the limit which the mean velocity of the particle for a period of time immediately succeeding the instant in question approaches as the period of time is allowed to decrease without limit.

247. Suppose that a particle moving on a known path passes over a distance s in a time t. To find the magnitude and direction of the particle at any given instant.

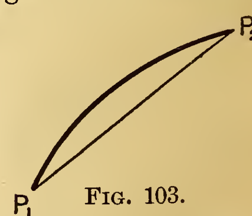


FIG. 103.

Let P_1 be the position of the particle at the instant in question, and P_2 its position at the end of a time t.

Denote the chord P_1P_2 by Δc, and the arc P_1P_2 by Δs. Since s increases as t increases, both Δc and Δs are positive.

By definition, $\dfrac{\Delta c}{\Delta t}=$ average velocity of the particle for a time Δt immediately succeeding the instant in question, and $\underset{\Delta t \doteq 0}{\text{limit}}\left[\dfrac{\Delta c}{\Delta t}\right]=$ velocity, v, of the particle at the instant in question.

Now
$$\underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{\Delta c}{\Delta t}\right]=\underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{\Delta s}{\Delta t}\cdot\frac{\Delta c}{\Delta s}\right]$$
$$=\underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{\Delta s}{\Delta t}\right]\cdot\underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{\Delta c}{\Delta s}\right]$$
$$=\underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{\Delta s}{\Delta t}\right],\text{ since }\underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{\Delta c}{\Delta s}\right]=1,$$
$$=\frac{ds}{dt}.$$
$$\therefore\ v=\frac{ds}{dt}.$$

Therefore the velocity, v, of the particle at any given instant $=\dfrac{ds}{dt}$, where t has the value corresponding to the instant in question.

As $\Delta t \doteq 0$, the chord P_1P_2 approaches as its limit the tangent line to the curve at P_1. Therefore the direction of the velocity at a given instant is that of the tangent line to the curve at the point which the particle occupies at that instant.

248. In Chapter III, and the preceding article, we have supposed that s is the distance passed over by a particle in a time t. We shall find it convenient, however, to take s sometimes not as the distance passed over by the particle in a given time but as a constant, *minus* this distance.

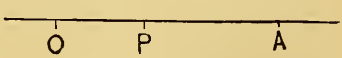


Fig. 104.

Thus, if the particle moves in a time t from a point A to a point P under a force directed towards a fixed point O and varying as some power of the distance of the particle from O, it will be convenient, as we shall see later, to take OP, not AP, equal to s. Then $AP = OA - s$.

When the distance passed over by a particle in a given time is taken as a constant, *minus* s, then $v = -\frac{ds}{dt}$. In this case s decreases as t increases.

249. The magnitude of a velocity is the magnitude of the displacement during the unit of time, or the magnitude of the displacement if the particle moved with uniform velocity during the unit of time. The direction of a velocity is the direction of the displacement in the unit of time, or the limit which this displacement for a period of time approaches as the period of time is allowed to decrease without limit. Then, since velocity has a definite magnitude and a definite direction, it follows that velocities may be compounded or resolved into component velocities in the same manner as displacements.

250. When the velocities of a particle at two points are known, the change in velocity can be found as follows:

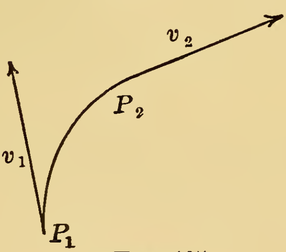


FIG. 105.

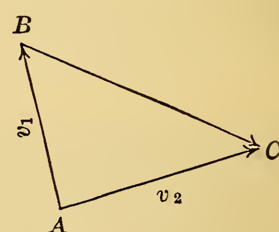


FIG. 106.

Let v_1 represent in magnitude and direction the velocity at P_1, and v_2 the velocity at P_2 (Fig. 105). At any point A (Fig. 106), draw AB and AC equal and parallel to v_1 and v_2 respectively, with the arrowheads towards B and C.

The sum of v_1 and BC is v_2. Therefore BC represents in magnitude and direction the change in velocity of the particle between P_1 and P_2.

Change in velocity must be distinguished from change in speed.

Thus, the change in speed in the above illustration is the length of v_2, *minus* the length of v_1.

251. Definitions. The **mean acceleration** of a moving particle in a given time is a quantity which has for direction the direction of the change in velocity during that time, and for magnitude the magnitude of the change in velocity divided by the time.

Thus, in the illustration of Art. 250, if t is the given time, $\frac{BC}{t}$ is the mean acceleration.

The **acceleration** of a particle at a given instant is the limit which the mean acceleration of the particle for a period of time immediately succeeding the instant in question approaches as the period of time is allowed to decrease without limit.

If a particle moves in such a manner that the acceleration at every instant is the same in magnitude and direction, it is said to move with **uniform acceleration.**

252. The magnitude of an acceleration is the magnitude of the change in velocity during the unit of time, or the magnitude of the change in velocity if the particle moved with uniform acceleration during the time. The direction of an acceleration is the direction of the change in velocity during the unit of time, or the limit which the change in velocity approaches as the period of time decreases without limit. Then, since acceleration has a definite magnitude and a definite direction, it follows that accelerations may be compounded or resolved into component accelerations in the same manner as velocities or displacements.

253. To find the magnitude and direction of an acceleration at any instant in terms of the magnitude and direction of the velocity at that instant:

Let P_1 be the position of the particle at the instant in question (see Fig. 107). Let P_2 be the position of the particle after a time Δt. Let α_N be the acceleration along the normal line and α_T the acceleration along the tangent line, at P_1.

Let P_1R_1 and P_2R_2 represent in magnitude and direction the velocities v and $v + \Delta v$ at P_1 and P_2 respectively. At P_1 draw P_1S equal and parallel to P_2R_2. Join R_1S. From S drop perpendiculars on the normal and tangent lines at P_1 to meet these lines in Q and T respectively.

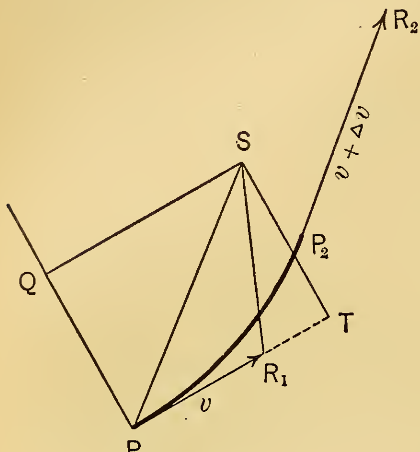


FIG. 107.

The line R_1S represents in magnitude and direction the change in velocity, Δv, during the time Δt.

Denote the arc P_1P_2 by Δs.

At first suppose that s increases as t increases, so that Δs is positive.

The projection P_1Q of R_1S on the normal line is the change in velocity along the normal line during the time Δt. Then $\underset{\Delta t \doteq 0}{\text{limit}} \left[\frac{P_1Q}{\Delta t}\right]$ is the acceleration along the normal line at P_1, or α_N.

Denote the angle between P_1R_1 and P_2R_2 by $\Delta\phi$.

$$\text{Then } \alpha_N = \underset{\Delta t \doteq 0}{\text{limit}} \left[\frac{P_1Q}{\Delta t}\right] = \underset{\Delta t \doteq 0}{\text{limit}} \left[\frac{P_1S \sin \Delta\phi}{\Delta t}\right]$$

$$= \underset{\Delta t \doteq 0}{\text{limit}} \left[(v + \Delta v)\frac{\sin \Delta\phi}{\Delta\phi} \cdot \frac{\Delta\phi}{\Delta s} \cdot \frac{\Delta s}{\Delta t}\right]$$

$$= v \cdot 1 \cdot \frac{1}{\rho} \cdot v$$

$$= \frac{v^2}{\rho},$$

where ρ is the radius of curvature of the curve at P_1.

The projection R_1T of R_1S on the tangent line is the change in velocity along the tangent line during the time Δt. Then $\underset{\Delta t \doteq 0}{\text{limit}} \left[\frac{R_1T}{\Delta t}\right]$ is the acceleration along the tangent line at P_1, or α_T.

Then

$$\alpha_T = \underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{R_1T}{\Delta t}\right] = \underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{P_1S\cos\Delta\phi - P_1R_1}{\Delta t}\right]$$

$$= \underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{(v+\Delta v)\cos\Delta\phi - v}{\Delta t}\right]$$

$$= \underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{v(\cos\Delta\phi - 1) + \Delta v\cos\Delta\phi}{\Delta t}\right]$$

$$= \underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{2\,v\sin^2\frac{1}{2}\Delta\phi}{\Delta t} + \frac{\Delta v}{\Delta t}\cos\Delta\phi\right]$$

$$= \underset{\Delta t \doteq 0}{\text{limit}}\left[\frac{\sin\frac{1}{2}\Delta\phi}{\frac{1}{2}\Delta\phi}\cdot v\sin\tfrac{1}{2}\Delta\phi\cdot\frac{\Delta\phi}{\Delta t} + \frac{\Delta v}{\Delta t}\cos\Delta\phi\right]$$

$$= 1\cdot 0\cdot\frac{d\phi}{dt} + \frac{dv}{dt}$$

$$= \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

The resultant acceleration is therefore

$$= \sqrt{\alpha_N^2 + \alpha_T^2}$$

$$= \sqrt{\frac{v^4}{\rho^2} + \left(\frac{dv}{dt}\right)^2},$$

and has the direction $\theta = \tan^{-1}\dfrac{v^2}{\rho\dfrac{dv}{dt}}$, where θ is the angle between the tangent to the curve at the point P_1 and the direction of the resultant acceleration.

If s decreases as t increases, α_N is as before, while $\alpha_T = -\dfrac{d^2s}{dt^2}$.

If motion is along a straight line, $\dfrac{v^2}{\rho}$, or α_N, is zero. Then in motion along a straight line, the magnitude of the acceleration α is $\dfrac{d^2s}{dt^2}$ or $-\dfrac{d^2s}{dt^2}$, the $+$ or $-$ sign being taken according as s increases or decreases as t increases, and the direction of the acceleration is in the direction of the motion.

EXERCISES

1. Show that the resultant P_1P_n of any number of component displacements $P_1P_2, P_2P_3, \cdots, P_{n-1}P_n$ is the diagonal of a rectangle P_1AP_nB in which P_1A and P_1B are the sums of the displacements resolved along the lines P_1A and P_1B respectively, which are perpendicular to each other.

2. A body undergoes three component displacements, 30 feet N., 60° E.; 40 feet S.; 50 feet W., 30° N. Find the resultant displacement. *Ans.* $10\sqrt{3}$ ft. W.

3. A wheel 4 feet in diameter rolls along a horizontal road, turning through an angle of 30°. Find the displacement of the point of the wheel initially in contact with the road relative to the point on the road with which it was in contact.

Ans. 0.272 ft., inclined 80° (approx.) to the horizontal.

4. A ship is carried by her screw 12 miles N., by the wind 4 miles E., 15° N., and by the current 2 miles S., 15° E. Find the component in a northeasterly direction of the resultant displacement. *Ans.* 10.95 mi.

5. A ship is influenced by her screw, the wind, and a current. Her screw carries her 12 miles N., the wind carries her 3 miles S., 15° E., and it is observed that her total displacement is 15 miles in a northeasterly direction. Find her displacement due to the current and the direction of the current.

Ans. 9.94 mi. E., 8° 42′ N.

6. A particle moves 100 feet W., 30° N. in 30 seconds, and thence 50 feet S., 60° W. in 20 seconds. Find (*a*) the mean speed; (*b*) the mean velocity of the particle during its time of motion.

Ans. (*a*) 3 ft. per sec.; (*b*) 2.645 ft. per sec. W., 10° 53.6′ N.

7. An automobile is running north at the rate of 16 miles per hour. A carriage going at the rate of 8 miles per hour appears to a person in the automobile to be going at the rate of 20 miles per hour. Find the direction of the carriage.

Ans. 71° 47.4′ W. or E. of S.

8. A train is running at the rate of 50 miles per hour in a shower of rain. If the drops fall vertically with a speed of 200 feet per second, in what direction will they seem to a man on the train to fall?

Ans. At an angle of $20° \, 8'.2$ with the vertical.

9. A river has a current which runs at the rate of a miles per hour. A steamer whose speed is b miles per hour in still water is to be run straight across. Find the direction in which she must be steered. *Ans.* At $\cos^{-1}\frac{a}{b}$ with the river bank.

10. A train running at the rate of 50 miles per hour is hit by a stone moving horizontally and at right angles to the track, at the rate of 30 feet per second. Find the magnitude of the velocity with which the stone hits the train, and the angle which it makes with the direction of motion of the train.

Ans. 79.2 ft. per sec.; $\tan^{-1}\frac{9}{22}$.

11. The initial and final velocities of a body which moves for 3 hours is 10 miles per hour E., 30° N., and 5 miles per hour N. respectively. Find the mean acceleration of the body.

Ans. $\frac{5}{3}\sqrt{3}$ mi.-per-hr. per hr. W.

12. A particle moves with uniform speed along the circumference of a circle of radius 6 feet from one end of a diameter to the other in 10 seconds. Find (a) the mean speed, (b) the mean velocity, of the particle for the time of motion.

Ans. 1.88 ft. per sec.; (b) 1.2 ft. per sec. in the direction of the diameter.

13. In Exercise 12, show that the acceleration is directed towards the center of the circle and is 0.592 ft.-per-sec. per sec.

14. A horse travels with uniform speed on an elliptical race track, semi-axes 800 and 400 feet respectively, from one end of the major axis to the other in 1 minute 25 seconds. Find (a) the mean speed; (b) the mean velocity of the horse during the time of motion. *Ans.* 15.52 mi. per hr.; 12.83 mi. per hr. in the direction of the major axis.

CHAPTER XXXI

FORCE. MASS. EQUATION OF MOTION OF A PARTICLE

254. Our idea of force is ultimate, like that of color, taste, and smell, and like them cannot be described. We all have the idea of force, however. When one speaks of exerting force we know exactly what he means.

255. Let us first choose as the unit of force that force exerted by a given spring when stretched by a given amount. Suppose that we have several exactly similar springs. Then if they are stretched by the same amount, they exert the same forces. Now suppose that we attach one of these springs to a body lying on a smooth table, and keeping it always stretched by the given amount and in such a position that the force exerted by it acts in a horizontal straight line, allow it to act on the body during known periods of time. By noting the distances through which the body moves, we are able to determine the magnitude and direction of the acceleration produced. The same determinations may be made with two or more springs. These experiments, although of necessity rough, will nevertheless be sufficient to suggest the following fundamental principles of motion: —

(1) In all cases the acceleration is uniform.

(2) In all cases the direction of the acceleration is that of the force.

(3) The acceleration produced by the force is proportional to the force, double the force producing double the acceleration, three times the force three times the acceleration, and so on.

(4) The accelerations do not depend on the lengths of time during which the forces act.

256. The first of the results of the preceding article shows that the acceleration is the same whether the body starts from rest or with an initial velocity. The third shows that if the force is zero, the acceleration is zero, and that therefore the body, if initially at rest, remains at rest, and if initially in motion, remains in motion with a velocity equal to the initial velocity.

257. If we indicate the magnitudes of the force and acceleration by F and α respectively, the third of the results of Art. 255 gives $F \propto \alpha$.

If $F \propto \alpha$, $\dfrac{F}{\alpha}$ is constant.

258. **Definition.** We shall define the **mass** of the body by saying that it is a number proportional to the constant ratio $\dfrac{F}{\alpha}$.

Thus, if m denotes the mass of a body, $m \propto \dfrac{F}{\alpha}$.

If $\dfrac{F}{\alpha} \propto m$, $\dfrac{F}{\alpha} = km$, where k is a constant depending on the units chosen for F, m, and α.

259. So far we have supposed that the forces remain constant both in magnitude and direction during the whole time of the motion. Let us now suppose that they vary in magnitude or direction, or in both magnitude and direction. Suppose that at a particular time t the force acting is F, and the acceleration produced by the force is α. If F remained constant in magnitude and direction for a period of time Δt immediately succeeding t_0, the equation $F = km\alpha$ would hold true for that time. Then, since the equation is independent of the time of motion, it would hold true if Δt be taken as small as we please. From this we conclude that the limit, as Δt approaches zero, of $F = km\alpha$ is $F = km\alpha$. That is, the variable force F at any time t_0 is such that $F = km\alpha$, where α is the acceleration of the particle at the time t_0, and has the direction of the force at that time.

Definition. The equation $F = kma$ is called the **equation of motion** of a particle.

260. Let F and F' be two forces which, acting on a body of mass m, produce accelerations of a and a' respectively.

Then $$F = kma,$$

and $$F' = kma'.$$

$$\therefore\ F : F' :: a : a'.$$

Then to compare two forces we may allow them to act on the same body and compare the two accelerations.

Let F be a force which, acting on two bodies of masses m and m', produces the accelerations a and a' respectively.

Then $$F = kma,$$

and $$F = km'a'.$$

$$\therefore\ m : m' :: a' : a.$$

Then to compare the masses of two bodies we may allow the same force to act on the bodies and compare the accelerations.

261. **Definition.** The force which the earth exerts on a body in the neighborhood of the earth is called the **weight** of the body.

262. We saw in Art. 260 how force and mass can be measured. It remains to choose units for them.

It is desirable to choose units for them so that k will have the value 1. We shall, therefore, assign a unit arbitrarily to either the force or mass, and define the other unit so that unit of force, acting on unit of mass, produces unit of acceleration.

263. There are two systems of units in common use: the **Gravitational System** and the **Absolute System.**

In the Gravitational System, the unit of force is chosen arbitrarily and the unit of mass is defined to be that mass which acted upon by unit of force produces unit of acceleration.

In the Absolute System, the unit of mass is chosen arbitrarily, and the unit of force is defined to be that force which acting on unit mass produces unit of acceleration.

264. In the Gravitational System of units, we may use the foot-pound-second (F.P.S.) System, or the centimeter-gram-second (C.G.S.) System.

In the F.P.S. Gravitational System, the unit of force is defined to be the weight of a certain piece of platinum, called the pound, stored in the Standards Office in London. The unit of acceleration is 1 foot-per-second per second. To determine the unit of mass: The weight of the pound produces in its mass an acceleration of g feet-per-second per second. Then, since the masses are inversely proportional to the accelerations (see Art. 260), the unit of mass must be g times the mass of the pound.

In this system of units, therefore,

$$F = ma,$$

where F indicates the force expressed in pounds weight, m the mass expressed in terms of a unit which is g times the mass of the pound, and a the acceleration in feet-per-second per second.

In the C.G.S. Gravitational System, the unit of force is the weight of the one-thousandth part of a piece of platinum called the kilogram stored in the Palais des Archives in Paris. The unit of acceleration is the centimeter-per-second per second. To determine the unit of mass: The weight of the gram produces in a gram mass an acceleration of g' centimeters-per-second per second, where $g' = g$ times the length of the foot in centimeters. Then the unit of mass must be g' times the mass of the gram.

In this system of units, therefore,

$$F = ma,$$

where F indicates the force expressed in grams weight, m

the mass expressed in terms of a unit which is g' times the mass of the gram, and a the acceleration in centimeters-per-second per second.

265. **In the F.P.S. Absolute System,** the unit of mass is defined to be the mass of the pound. The unit of force will therefore be that force which acting on the unit of mass produces an acceleration of 1 foot-per-second per second. The weight of the pound produces in the unit of mass an acceleration of g feet-per-second per second. Then, since forces are proportional to the accelerations, the force that produces the acceleration of 1 foot-per-second per second in the mass of the pound is $\frac{1}{g}$.

Definition. The force that produces in the mass of the pound an acceleration of 1 foot-per-second per second is called the **poundal.**

The poundal is $\frac{1}{g}$ times the weight of the pound, or the weight of about half an ounce.

In this system of units, therefore,

$$F = ma,$$

where F indicates the force in poundals, m the mass in pounds, and a the acceleration in feet-per-second per second.

In the C.G.S. Absolute System, the unit of mass is the mass of the one-thousandth part of the kilogram. The unit of force is the force which produces in the unit of mass an acceleration of 1 centimeter-per-second per second. This force is $\frac{1}{g'}$ times the weight of the gram.

Definition. The force which produces in the mass of the gram an acceleration of 1 centimeter-per-second per second is called the **dyne.**

The dyne is $\frac{1}{g'}$ times the weight of the gram, or the weight of $\frac{1}{981}$ of a gram (nearly)

In this system of units, therefore,

$$F = ma,$$

where F indicates the force in dynes, m the mass in grams, and a the acceleration in centimeters-per-second per second.

266. The numerical relation between the Gravitational and the Absolute systems of units are:

The weight of 1 gram $= g'$ dynes.

The weight of 1 pound $= g$ poundals.

267. **Definition.** The product of the mass of a body and its velocity in any given direction is called the **momentum** of the body in that direction. It is also sometimes called the **quantity of motion** of the body, or merely, the **motion** of the body.

Since the momentum is mv, the rate of change of momentum is ma.

268. The results of a physical experiment are at best more or less inaccurate, and consequently a law derived from experiments should be subjected to all possible tests before being accepted as true. The experiments mentioned in Art. 255 as leading to the law of motion $F = ma$ are so extremely rough as to be of no value in the determination of this law. They merely serve to suggest that such a law may be possible. Then, before this law is accepted as the principle governing motion, it should be subjected to further tests. Such tests have been made in a number of ways, and in all cases the law has been found satisfactory. One of the most convincing tests is that made continually by astronomers, when, by assuming this law, they find, by direct observation and mathematical reasoning, the exact positions which heavenly bodies will occupy at certain times.

269. Without entering further into the discussion, we shall assume the truth of the law that $F = ma$ and shall also assume

as true the law first enunciated by Galileo, and therefore bearing his name, which is expressed by Thompson and Tait in the following words:[1]

"When any forces whatever act on a body, then, whether the body be originally at rest or moving with any velocity in any direction, each force produces in the body the exact change of motion which it would have produced if it had acted singly on the body originally at rest."

270. Since, by Galileo's law, each of several forces acting on a body produces the same momentum and therefore the same rate of change of momentum as if it acted singly on the body, it follows that forces may be compounded or resolved into components in the same manner as accelerations and therefore as velocities or displacements.

EXERCISES

In the exercises of this chapter and the following chapters, assume that

$$1 \text{ pound avoirdupois} = 453.59 \text{ grams.}$$
$$1 \text{ meter} = 3.281 \text{ feet.}$$
$$g = 32.$$

1. Two forces produce in two masses, accelerations of 20 and 25 units respectively. Find (*a*) the ratio of the forces if the masses are equal; (*b*) the ratio of the masses if the forces are equal. *Ans.* (*a*) 4 : 5; (*b*) 5 : 4.

2. Compare the values for the mass of a body expressed in the ft.-pound-sec. and the yd.-ton-min. gravitational systems. *Ans.* $24 \times 10^5 : 1$.

3. Compare the values for the mass of a body expressed in the cm.-gr.-sec. and the meter-kilogram-min. systems. *Ans.* $36 \times 10^3 : 1$.

[1] See MacGregor, *Kinematics and Dynamics*, p. 204.

4. Convert 1 poundal to absolute units in the yd.-ton-min. system. *Ans.* $\frac{3}{5}$.

5. Convert 1 poundal to dynes. *Ans.* 13,824.7.

6. Convert 1 pound to dynes. *Ans.* 442,392.

7. Find the acceleration produced in a mass of 10 pounds by a force of 20 dynes.

Ans. 0.000145 ft.-per-sec. per sec. (approx.).

8. Find the force expressed in pounds which produces in a mass of 20 pounds an acceleration of 12 yd.-per-min. per second.

Ans. $\dfrac{12}{g}$.

9. Find the mass on which a force of 12 pounds produces an acceleration of 20 ft.-per-sec. per sec. *Ans.* $\dfrac{3g}{5}$ lb.

10. Find the resultant of two forces of 3 and 5 pounds respectively, acting on a particle at an angle of 60°.

Ans. The resultant is a force of 7 lb. acting at an angle of about 21° 47′ with the force of 5 lb.

11. Forces of 2, 3, and 5 pounds respectively acting on a particle and in one plane make angles of 120° with one another. Find the resultant of the forces.

Ans. The resultant is a force of $\sqrt{7}$ lb. acting at an angle of about 19° 6′ with the force of 5 lb. and about 40° 54′ with the negative direction of the force of 2 lb.

12. Three equal forces, P, acting on a particle and in one plane make angles of 120° with one another. Find the resultant of the forces. *Ans.* The resultant is zero.

13. A man weighs 160 pounds. Find his apparent weight when in an elevator which descends with an acceleration of 2 ft.-per-sec. per sec. *Ans.* 150 lb.

14. A man who weighs 160 pounds notices that in an elevator his apparent weight is 180 pounds. Show that the elevator has an upward acceleration of 4 ft.-per-sec. per sec.

CHAPTER XXXII

RECTILINEAR MOTION

In this chapter, we shall consider the force or forces acting on the particle to be such that motion takes place in a straight line.

MOTION UNDER A CONSTANT FORCE

271. The simplest case of motion is that of a particle moving under a constant force.

Suppose that a particle of mass m is acted upon by a constant force f in the direction of its motion. To determine the motion of the particle.

Take O (Fig. 108), the initial position of the particle, as the origin, and the direction of motion as the positive direction of the axis.

FIG. 108.

Let s be the abscissa of the particle at any time t.

We shall suppose that m, f, and a are expressed in terms of the unit of mass, force, and acceleration respectively, in some chosen system of units.

Then $$ma = f.$$

Since s increases as t increases, $a = \frac{d^2s}{dt}$.

$$\therefore m\frac{d^2s}{dt^2} = f,$$

or $$\frac{d^2s}{dt^2} = \frac{f}{m}. \tag{1}$$

To express v and s in terms of t, we may proceed as follows:

Since
$$\frac{d^2s}{dt^2}=\frac{f}{m},$$

$$\therefore \frac{ds}{dt}=\frac{f}{m}t+c, \text{ by integration.}$$

To determine c: When $t=0$, $\frac{ds}{dt}=$ the initial velocity. Call it v_0.

$$\therefore \quad v_0=0+c. \qquad \therefore \quad c=v_0.$$

$$\therefore \quad \frac{ds}{dt}=\frac{f}{m}t+v_0.$$

$$\therefore \quad v=\frac{f}{m}t+v_0. \tag{2}$$

By a second integration, we get

$$s=\tfrac{1}{2}\frac{f}{m}t^2+v_0t+c_1. \tag{3}$$

To determine c_1: When $t=0$, $s=$ the space passed over before we begin to count time. In this case, therefore, $s=0$ when $t=0$.

$$\therefore \quad s=\tfrac{1}{2}\frac{f}{m}t^2+v_0t.$$

To determine v in terms of s, we may express t in terms of v from (2), substitute the result in (3), and solve the resulting equation for v. It is more convenient, however, to proceed as follows:

Multiply (1) by $2\frac{ds}{dt}$.

$$\therefore \quad 2\frac{ds}{dt}\left(\frac{d^2s}{dt^2}\right)=2\frac{f}{m}\frac{ds}{dt}.$$

Now
$$2\frac{ds}{dt}\left(\frac{d^2s}{dt^2}\right)=\frac{d}{dt}\left(\frac{ds}{dt}\right)^2.$$

$$\therefore \quad \frac{d}{dt}\left(\frac{ds}{dt}\right)^2=2\frac{f}{m}\frac{ds}{dt}.$$

Integrate. $$\therefore \left(\frac{ds}{dt}\right)^2 = 2\frac{f}{m}s + c.$$

$$\therefore v^2 = 2\frac{f}{m}s + c.$$

To determine c: when $s = 0$, $v = v_0$. $\therefore c = v_0^2$.

$$\therefore v^2 = 2\frac{f}{m}s + v_0^2.$$

272. As an illustration of motion under a constant force, consider the following example:

A particle of mass m, initially at rest, slides down a smooth inclined plane of length a, which makes an angle ϕ with the horizontal. To determine the velocity of the particle at the foot of the plane, and the time which the particle takes to reach the foot of the plane.

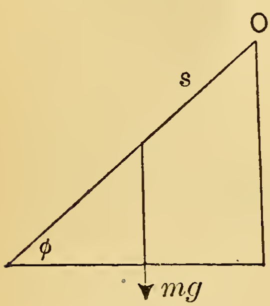


FIG. 109.

Take O (Fig. 109), the initial position of the particle, as the origin, and the direction of motion as the direction of the axis.

The only force tending to produce motion is the component weight of the particle acting along the plane. This force is $mg \sin \phi$, where g depends on the unit system chosen.

The equation of motion is therefore

$$m\frac{d^2s}{dt^2} = mg \sin \phi,$$

or
$$\frac{d^2s}{dt^2} = g \sin \phi. \tag{1}$$

To determine the velocity at the foot of the plane, multiply (1) by $2\frac{ds}{dt}$, integrate, and determine the constant of integration.

$$\therefore v^2 = 2\,gs \sin \phi. \tag{2}$$

Therefore, at the foot of the plane,

$$v^2 = 2\,ga \sin\phi, \text{ or } v = \sqrt{2\,ga \sin\phi}.$$

To determine the time which the particle takes to reach the foot of the plane, we may proceed as in the last article, or we may integrate (2) directly. Choosing the latter way, we have

$$\frac{ds}{dt} = \sqrt{2\,gs \sin\phi},$$

$$= \sqrt{2\,g \sin\phi} \cdot \sqrt{s}.$$

Separate the variables.

$$\therefore\ \frac{ds}{\sqrt{s}} = \sqrt{2\,g \sin\phi}\, dt.$$

Integrate, and determine the constant of integration.

$$\therefore\ 2\sqrt{s} = \sqrt{2\,g \sin\phi}\, t.$$

$$\therefore\ s = \tfrac{1}{2}\, g \sin\phi\, t^2.$$

ATTRACTIVE FORCE VARYING DIRECTLY AS THE DISTANCE

273. Suppose that a particle of mass m, initially at rest, is attracted towards a fixed point with a force varying directly as the distance. To determine the motion of the particle.

Let O (Fig. 110), the position of the fixed point, be the origin, and A, distant a from O, the initial position of the particle. Let the opposite direction to the motion at the start be the positive direction of the axis. Let s be the abscissa of the particle at any time t.

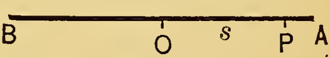


Fig. 110.

We shall suppose as before that m, force, and a are expressed in terms of the units of mass, force, and acceleration respectively, in some chosen system of units.

The equation of motion of the particle between A and O is

$$ma = ks, \tag{1}$$

where k is a constant at present undetermined.

To determine k, we must know the magnitude of the attractive force or acceleration at some point of the path. Suppose for definiteness that the magnitude of the acceleration at unit distance from O is known. Suppose that it is μ. Then $m\mu = k \cdot 1$. $\therefore k = m\mu$. Equation (1) is therefore

$$ma = m\mu s,$$

or

$$a = \mu s. \tag{2}$$

Since s decreases as t increases, $a = -\frac{d^2s}{dt^2}$.

Equation (2) is therefore
$$\frac{d^2s}{dt^2} = -\mu s. \tag{3}$$

Equation (3) is the equation of motion of the particle from A to O.

Multiply (3) by $2\frac{ds}{dt}$, integrate, and determine the constant of integration.

$$\therefore \left(\frac{ds}{dt}\right)^2 = \mu(a^2 - s^2).$$

$$\therefore \frac{ds}{dt} = -\sqrt{\mu}\sqrt{a^2 - s^2}, \tag{4}$$

the minus sign being taken because $\frac{ds}{dt}$ is negative, since s decreases as t increases.

Separate the variables in (4).

$$\therefore \frac{ds}{\sqrt{a^2 - s^2}} = -\sqrt{\mu}\, dt.$$

Integrate, and determine the constant of integration.

$$\therefore \sin^{-1}\frac{s}{a} = -\sqrt{\mu}\, t + \frac{\pi}{2}.$$

$$\therefore s = a \sin\left(-\sqrt{\mu}\, t + \frac{\pi}{2}\right).$$

$$\therefore s = a \cos \sqrt{\mu}\, t. \tag{5}$$

Differentiate (5) with respect to t.

$$\therefore \frac{ds}{dt} = -a\sqrt{\mu} \sin \sqrt{\mu}\, t. \tag{6}$$

Equations (4), (5), and (6) give the relations between each pair of the variables v, s, and t.

From equation (4) we see that when $s=0$, $\frac{ds}{dt}=-a\sqrt{\mu}$. Since the particle has therefore a velocity at O and there is no force acting on it there, it will move past O. At any distance s to the left of O there is a force the same in magnitude but opposite in direction acting on it as acted at the distance s to the right. We see therefore that the motion will be checked as rapidly as it was increased, and that the particle will move to a point B, such that $OB=-a$, in the same time as it took to reach O from A. At B the same force is acting on the particle as acted before at A. The particle will therefore return to A, moving exactly as it did before, and continue oscillating between A and B in equal times.

From equation (5), we see that when $s=0$, $t=\frac{\pi}{2\sqrt{\mu}}$. The time taken for the particle to return to A is therefore $t=\frac{2\pi}{\sqrt{\mu}}$. This result is remarkable inasmuch as it is independent of the initial distance of the particle from O, and depends only on the intensity of the attraction at O.

274. Definition. When a body oscillates between two points, the time of a complete oscillation is called the **periodic time** of the body.

In the above example the periodic time is $T=\frac{2\pi}{\sqrt{\mu}}$.

275. On BA of Fig. 110 as diameter describe a circle (see Fig. 111). From P, the position of the particle at a time t, draw a perpendicular PM to meet the circle in M. Denote the angle POM by θ.

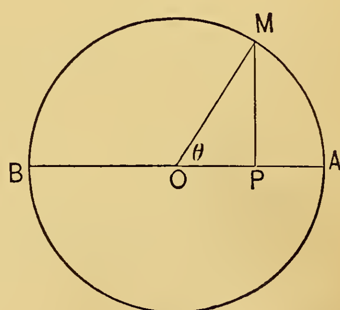


FIG. 111.

Therefore $\quad s=a\cos\theta$.

Also $\quad s=a\cos\sqrt{\mu}\,t$.

$$\therefore\ \theta=\sqrt{\mu}\,t. \tag{1}$$

Equation (1) shows that if a particle is acted upon by an attractive force varying directly as the distance from a fixed point, its position P at any time t is the projection on the line BA of the point M, which moves with the uniform angular velocity $\sqrt{\mu}$ on the circumference of the circle AMB.

276. **Definition.** If a body moves in a straight line in such a way that its equation of motion is $\frac{d^2s}{dt^2} = -\mu s$, where μ is a positive constant, or can be reduced to this form, it is said to have **simple harmonic motion.**

277. It has been determined that the tension in a stretched elastic string varies directly as the extension of the string beyond its natural length. This principle is known as Hooke's Law.

From Hooke's Law it follows that, if a particle moves under the force exerted on it by a stretched elastic string, the motion is simple harmonic motion.

ATTRACTIVE FORCE VARYING INVERSELY AS THE SQUARE OF THE DISTANCE

278. Suppose that a particle of mass m, initially at rest, is attracted towards a fixed point with a force varying inversely as the square of the distance. To determine the motion of the particle.

Let O (Fig. 112), the position of the fixed point, be the origin, and A distant a from O, the initial position of the particle. Let the opposite direction to the motion at the start be the positive direction of the axis. Let s be the abscissa of the particle at any time t.

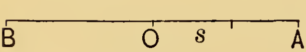


Fig. 112.

With properly chosen units for mass, force, and acceleration, the equation of motion of the particle between A and O is

$$ma = \frac{k}{s^2}. \tag{1}$$

To determine k: Suppose that the magnitude of the acceleration at unit distance from O is μ.

Equation (1) is therefore $ma = \dfrac{m\mu}{s^2}$,

or
$$a = \frac{\mu}{s^2}. \tag{2}$$

Since s decreases as t increases, $a = -\dfrac{d^2s}{dt^2}$.

Equation (2) is therefore
$$\frac{d^2s}{dt^2} = -\frac{\mu}{s^2}. \tag{3}$$

Equation (3) is the equation of motion of the particle between A and O.

Multiply (3) by $2\dfrac{ds}{dt}$, integrate, and determine the constant of integration.

$$\therefore \left(\frac{ds}{dt}\right)^2 = 2\mu\left(\frac{1}{s} - \frac{1}{a}\right).$$

$$\therefore \frac{ds}{dt} = -\sqrt{2\mu}\sqrt{\left(\frac{1}{s} - \frac{1}{a}\right)}, \tag{4}$$

the minus sign being taken because $\dfrac{ds}{dt}$ is negative, since s decreases as t increases.

Separate the variables in (4), integrate, and determine the constant of integration.

$$\therefore t = \sqrt{\frac{a}{2\mu}}\left[a\cos^{-1}\sqrt{\frac{s}{a}} + \sqrt{as - s^2}\right].$$

As the particle approaches O the force acting on it increases without limit. Also, as seen from (4), the velocity of the particle increases without limit. If we assume that when the particle is at O the force acting on it is zero, the particle will move past O. At any distance s to the left of O there is a force the same in magnitude but opposite in direction acting on it as acted at the distance s to the right. We can therefore, on this assumption, conclude, as in Art. 273, that the par-

ticle will move to a point B, such that $OB = -a$, in the same time as it took to reach O from A, and that it will oscillate between B and A in equal times.

In such a case the periodic time is $T = \pi a\sqrt{\dfrac{2a}{\mu}}$.

CASE OF MOTION IN A RESISTING MEDIUM

279. Suppose that a particle of mass m, initially at rest, moves under a constant force, f, in the direction of its motion, and in a resisting medium in which the resistance varies as the square of the velocity. To determine the motion of the particle.

Take O (Fig. 113), the initial position of the particle, as the origin, and the direction of the motion as the positive direction of the axis. Let s be the abscissa of the particle at any time t.

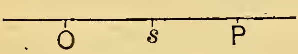


FIG. 113.

Since s increases as t increases, $\alpha = \dfrac{d^2s}{dt^2}$.

If the units of mass, force, and acceleration are properly chosen,

$$m\frac{d^2s}{dt^2} = f - k\left(\frac{ds}{dt}\right)^2. \tag{1}$$

To determine k, suppose that the velocity which the particle would need to have in order that the resistance might be equal to f is μ. Then, $f = k\mu^2$. $\therefore\ k = \dfrac{f}{\mu^2}$.

Equation (1) is therefore

$$\frac{d^2s}{dt^2} = \frac{f}{m\mu^2}\left[\mu^2 - \left(\frac{ds}{dt}\right)^2\right]. \tag{2}$$

It is immediately noticeable that equation (2) cannot be integrated by multiplication by $2\dfrac{ds}{dt}$. We can integrate, however, as follows:

Let $v = \frac{ds}{dt}$, and substitute in (2).

$$\therefore \frac{dv}{dt} = \frac{f}{m\mu^2}\left[\mu^2 - v^2\right]. \tag{3}$$

Separate the variables in (3), integrate, and determine the constant of integration.

$$\therefore \frac{1}{2\mu}\log\frac{\mu + v}{\mu - v} = \frac{f}{m\mu^2}t.$$

$$\therefore v = \mu\frac{e^{lt} - 1}{e^{lt} + 1}, \tag{4}$$

where $$l = \frac{2f}{m\mu}.$$

From equation (4), s can be determined immediately in terms of t by substituting $\frac{ds}{dt}$ for v, and integrating.

EXERCISES

Assume the resistance of the air zero unless otherwise specified.

1. A body is projected vertically downwards from the top of a tower 200 feet high with a velocity of 10 feet per second. Find the time which it takes to reach the ground and its velocity as it reaches the ground.

Ans. 3.24 secs.; 113.7 ft. per sec.

2. The same as exercise 1, excepting that the body is projected vertically upwards instead of vertically downwards.

Ans. 3.86 secs.; 113.7 ft. per sec.

3. A balloon is ascending with a velocity of 20 miles per hour. A man in the balloon drops a stone which reaches the ground in 6 seconds. Find the height of the balloon when the stone is dropped. *Ans.* 400 ft.

4. A particle of mass m is projected up a smooth plane inclined at an angle ϕ with the horizontal, with a velocity of v_0 feet per second. Find how high it will ascend the plane and the time it takes to return to its initial position.

Ans. $\dfrac{v_0^2}{2g\sin\phi}$ ft.; $\dfrac{2v_0}{g\sin\phi}$ secs.

5. Show that the times of descent of a body down all chords drawn from the highest point of a vertical circle are equal and the same as the time of falling down the vertical diameter.

6. A train, running at 15 miles per hour with steam shut off, strikes an up grade of 1 in 300. The resistance due to the air and friction is 8 pounds per ton. Find how far the train will run up the grade before coming to rest. *Ans.* 1031.25 ft.

7. In the preceding exercise, if the grade were a down grade, how far would the train run before coming to rest?

Ans. 11343.8 ft.

8. On the moon a pound weighs $2\frac{2}{3}$ ounces. On the earth a man can jump 5 feet high. How high could he jump on the moon? *Ans.* 30 ft.

9. A cable-car weighs 12 tons. Find the tension (assuming it uniform) in the cable if the car attains a velocity from rest of 12 feet per second in 40 seconds.

Ans. A force of 225 lb.

10. A train rounds a curve in the form of an arc of a circle of 500 feet radius at the rate of 30 miles per hour. Find the angle which the thread of a plummet suspended from the roof of the car makes with the vertical. *Ans.* $6° 54'$.

11. A car starts from rest with an acceleration of 4 ft.-per-sec. per sec. At what angle must a man brace himself to keep in equilibrium? *Ans.* $7° 8'$ (nearly).

12. A spiral spring of natural length 12 inches is stood on one end. A weight of 15 pounds placed on top of it will just hold it compressed to 9 inches. If the weight be placed on top of it when in its natural length and then released, determine the motion.

SUGGESTION. The spring is deformed in accordance with Hooke's Law.

Ans. The weight descends until the spring is compressed to 6 inches, ascends till the spring is of natural length, and keeps this motion up indefinitely.

$$T(\text{complete oscillation}) = \frac{2\pi}{\sqrt{128}} = .56 \text{ sec.}$$

13. A rubber string of natural length 2 feet is suspended by one end. A weight of $\frac{1}{2}$ pound attached to the other end will just hold the string stretched to $2\frac{1}{2}$ feet. If after the weight is attached the string be stretched to 3 feet and then released, determine the motion.

Ans. The weight ascends till the string is of natural length, descends till the string is stretched to 3 ft., and keeps this motion up indefinitely.

$$T = \frac{2\pi}{8} = .78 \text{ sec.}$$

14. A round spar of indefinite length, radius of cross section a, and mass M floats upright in water. After it comes to rest it is pushed down a distance of h feet and then released. Determine the motion.

Ans. It rises h ft. out of the water beyond mark when at rest, descends h ft. below mark when at rest, and keeps this motion up indefinitely.

$$T = \frac{2\pi}{\sqrt{k}} \text{ where } k = \frac{1000\,\pi a^2}{16\,M}.$$

15. A chain 64 feet long is hung over a smooth drum, one end being 2 feet lower than the other. Determine the velocity

of the chain when one end is 12 feet lower than the other, and the time taken to reach this velocity.

Ans. 5.92 ft. per second; 2.48 secs.

16. A spring which offers a resistance of 20 pounds per inch of compression stands upright on a fixed plane. A weight of 10 pounds falls 6 feet and strikes the spring. How far will the weight descend? *Ans.* $6\frac{3}{4}$ ft.

17. A ship, whose mass is 1000 tons, moves from rest under a constant force of 10 tons and is resisted by the water with a force proportional to the velocity and equal to 2000 poundals when the velocity is 1 foot per second. Find the acceleration, and the distance passed over, when the velocity is 10 miles per hour.

Ans. Accel. = 0.305 ft.-per-sec. per sec.; $S = 343$ ft. (approx.).

18. A weight of 160 pounds is placed on a plane inclined 20° to the horizontal. The resistance due to friction is 5.72 pounds. The resistance due to the wind varies as the square of the velocity, and is 2 pounds per square foot of surface when the velocity is 30 feet per second. Given that the area exposed to the wind is $4\frac{1}{2}$ square feet, find the velocity when the weight has descended 400 feet. Find the limiting value of the velocity on a hill of indefinite length. *Ans.* 62.54 ft.; 70 ft.

CHAPTER XXXIII

MOTION IN A PLANE CURVE

280. The simplest case of motion in a plane curve is that of a particle which has an initial velocity and is acted upon by a force, constant in magnitude, and in direction constant and oblique to that of the initial velocity. Such a case is that of a projectile, if the resistance of the air be neglected, because for distances through which a projectile carries, the force of gravity may be considered as constant both in magnitude and direction.

281. Suppose that a particle of mass m has an initial velocity v_0 which makes an angle α with the horizontal, and is acted upon by gravity alone. To determine the motion of the particle.

Take the initial position of the particle as the origin of coördinates. Take the axes vertical and horizontal. (Fig. 114.)

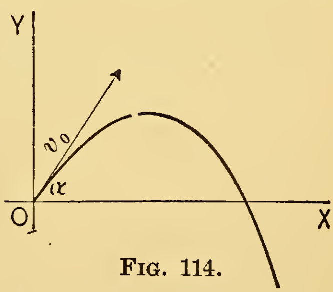


Fig. 114.

Since there are no forces acting on the particle in the direction of the x-axis,

$$\frac{d^2x}{dt^2}=0. \tag{1}$$

Since the force of gravity acts in the negative direction of the y-axis,

$$\frac{d^2y}{dt^2}=-g. \tag{2}$$

Integrate equation (1).

$$\therefore \frac{dx}{dt}=c.$$

To determine c: At O the component of the velocity along the x-axis is $v_0 \cos \alpha$.

$$\therefore\ c = v_0 \cos \alpha.$$

$$\therefore\ \frac{dx}{dt} = v_0 \cos \alpha. \tag{3}$$

Integrate equation (3), and determine the constant of integration.

$$\therefore\ x = v_0 \cos \alpha\ t. \tag{4}$$

Similarly, from equation (2), $y = -\frac{1}{2} gt^2 + v_0 \sin \alpha\ t.$ (5)

Substitute the value of t from (4) in (5).

$$\therefore\ y = -\frac{g}{2\, v_0^2 \cos^2 \alpha} x^2 + x \tan \alpha. \tag{6}$$

Equation (6) is the equation of the curve which the particle describes when moving under the above supposition. It is the equation of a parabola with its vertex at $\left(\frac{v_0^2 \sin 2\alpha}{2g}, \frac{v_0^2 \sin^2 \alpha}{2g}\right)$, and focus at $\left(\frac{v_0^2 \sin 2\alpha}{2g}, -\frac{v_0^2 \cos 2\alpha}{2g}\right)$.

MOTION OF A PARTICLE ON A SMOOTH VERTICAL CURVE

282. Let $y = f(x)$ be the equation of the curve on which the particle moves. Let s be the length of the arc between some fixed point A on the curve and the position P of the particle at the time t. Let ϕ be the angle which the tangent line to the curve at P makes with the x-axis.

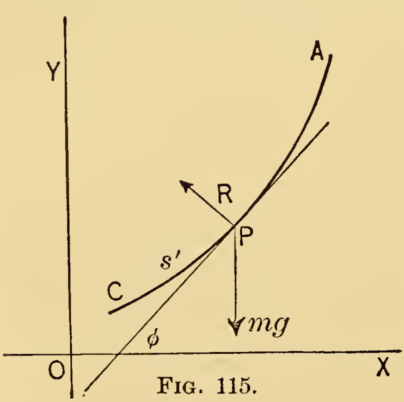


FIG. 115.

The forces acting on the particle are the force of gravity mg, acting vertically downwards, and the resistance R of the curve acting normal to the curve.

Since the resistance acts normal to the curve, it does not influence motion along the curve. The only force that contrib-

utes to motion along the curve is therefore the component of the force of gravity along the curve. The equation of motion is therefore

$$m\frac{d^2s}{dt^2} = -mg\sin\phi,$$

or

$$\frac{d^2s}{dt^2} = -g\sin\phi. \tag{1}$$

Now, $\sin\phi = \frac{dy}{ds}$, since $\tan\phi = \frac{dy}{dx}$.

$$\therefore\ \frac{d^2s}{dt^2} = -g\frac{dy}{ds}.$$

Multiply by $2\frac{ds}{dt}$.

$$\therefore\ \frac{d}{dt}\left(\frac{ds}{dt}\right)^2 = -2g\frac{dy}{dt}.$$

Integrate. $$\therefore\ \left(\frac{ds}{dt}\right)^2 = -2gy + c.$$

To determine c: Suppose that the body is at rest at a point A, and that the ordinate of A is h. Then $c = 2gh$.

$$\therefore\ \left(\frac{ds}{dt}\right)^2 = 2g(h-y). \tag{2}$$

To integrate further we must know y in terms of s, and therefore in terms of x. That is, we must know the equation of the curve.

283. Suppose that the curve of the preceding article is the inverted cycloid whose equations are:

$$x = a(\theta + \sin\theta),\ \ y = a(1-\cos\theta).$$

Then $dx = a(1+\cos\theta)\,d\theta$, and $dy = a\sin\theta\,d\theta$.

$$\therefore\ ds = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\,d\theta = 2a\cos\tfrac{1}{2}\theta\,d\theta.$$

Therefore, if s is measured from the origin,

$$s = 2a \int_0^{\theta} \cos \tfrac{1}{2}\theta \, d\theta = 4a \sin \tfrac{1}{2}\theta.$$

Since $\quad a(1-\cos\theta) = 2a\sin^2\tfrac{1}{2}\theta, \quad \therefore\ y = 2a\sin^2\tfrac{1}{2}\theta.$

$$\therefore\ y = \frac{s^2}{8a}.$$

Substitute this value for y in equation (2) of the preceding article.

$$\therefore\ \left(\frac{ds}{dt}\right)^2 = 2g\left(h - \frac{s^2}{8a}\right).$$

$$\therefore\ \frac{ds}{dt} = -\sqrt{2g}\sqrt{h - \frac{s^2}{8a}}.$$

Separate the variables, and integrate.

$$\therefore\ \sin^{-1}\frac{s}{\sqrt{8ah}} = -\sqrt{\frac{g}{4a}}\,t + c.$$

To determine c: When $t = 0$, $y = h$, and $\therefore\ s = \sqrt{8ah}$.
$\therefore\ c = \frac{\pi}{2}$.

$$\therefore\ \sin^{-1}\frac{s}{\sqrt{8ah}} = \frac{\pi}{2} - \sqrt{\frac{g}{4a}}\,t.$$

$$\therefore\ s = \sqrt{8ah}\sin\left(\frac{\pi}{2} - \sqrt{\frac{g}{4a}}\,t\right).$$

$$= \sqrt{8ah}\cos\sqrt{\frac{g}{4a}}\,t.$$

It can readily be shown that the particle will rise to a point B at a height h on the other side of the lowest point of the arch and oscillate between A and B in equal times, the periodic time being

$$T = 4\pi\sqrt{\frac{a}{g}}.$$

Since T does not involve h, it is independent of the position of A. Hence the periodic time is the same for all arcs of the cycloid. For this reason the curve is called the **tautochrone**.

THE CYCLOIDAL PENDULUM

284. As seen in Art. 117, the evolute of a cycloid is an equal cycloid having its vertices at the cusps of the given cycloid. Hence if a heavy particle be suspended from a point O (Fig. 116) by a string of length half the length of an arc of this cycloid, and the string in its oscillations be made to wrap itself on a solid piece in the shape of the evolute of the cycloid, the particle will describe the given cycloid.

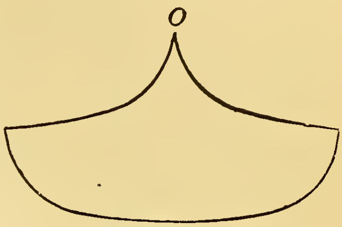


FIG. 116.

An arrangement of this sort is called the **cycloidal pendulum.**

If s and ϕ be measured as in Art. 282, the equation of motion for the cycloidal pendulum will be the same as in the case of a particle sliding down a smooth curve under the action of gravity. All the results of Art. 283 will therefore hold in the case of the cycloidal pendulum. We can thus conclude that the periodic time of a cycloidal pendulum for any length of arc through which the particle swings is $T = 4\pi\sqrt{\frac{a}{g}}$.

It is the property that the periodic time is independent of the length of the arc through which the particle swings that makes the cycloidal pendulum interesting.

THE SIMPLE PENDULUM

285. Suppose that a particle of mass m, suspended from a point by a string of length a, starts from the lowest point of its path with the velocity which it would acquire if it fell freely from rest in a vacuum through a distance h. To determine the motion of the particle.

Let C be the point from which the particle is suspended (Fig. 117). Let O, the lowest point of the arc of the circle through which the particle swings, be the origin of coördinates.

Let s be the distance from the origin to the position P of the particle at the time t. Let the coördinates of P be (x, y). Let ϕ be the angle which the tangent line to the curve at P makes with the x-axis.

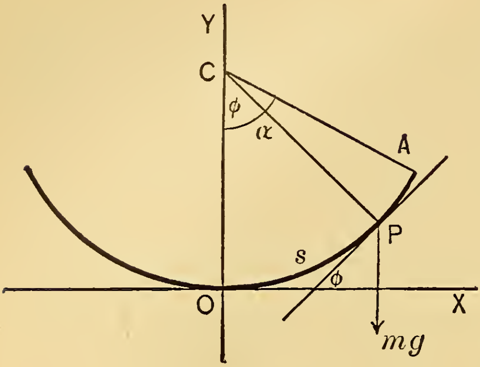


FIG. 117.

The force acting on the particle at a time t is $mg \sin \phi$. Therefore, since it acts opposite to the motion,

$$ma = -mg \sin \phi,$$

or
$$a = -g \sin \phi.$$

Since s increases as t increases, $a = \dfrac{d^2s}{dt^2}$.

$$\therefore \frac{d^2s}{dt^2} = -g \sin \phi.$$

Since
$$\sin \phi = \frac{dy}{ds}, \quad \therefore \frac{d^2s}{dt^2} = -g\frac{dy}{ds}. \tag{1}$$

Then, as in Art. 282,

$$\left(\frac{ds}{dt}\right)^2 = -2gy + c.$$

To determine c: The velocity which the particle would acquire in falling freely from rest in a vacuum through a distance h is $\sqrt{2gh}$. $\quad \therefore c = 2gh.$

$$\therefore \left(\frac{ds}{dt}\right)^2 = 2g(h - y). \tag{2}$$

The equation of the circle is $x^2 + y^2 - 2ay = 0$.

$$\therefore ds = \frac{a}{x}dy.$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = \frac{a^2}{x^2}\left(\frac{dy}{dt}\right)^2 = \frac{a^2}{2ay - y^2}\left(\frac{dy}{dt}\right)^2. \tag{3}$$

Substitute the result of (3) in (2).

$$\therefore \frac{dy}{dt} = \frac{\sqrt{2g}}{a}\sqrt{(h-y)(2ay-y^2)}.$$

$$\therefore t = \frac{a}{\sqrt{2g}}\int_0^y \frac{dy}{\sqrt{(h-y)(2ay-y^2)}} \tag{4}$$

is the time the particle takes to reach a point whose ordinate is y.

The expression for t in (4) is transformed into one or other of the forms

$$t = \sqrt{\frac{a}{g}}\int_0^x \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-\frac{h}{2a}x^2\right)}},$$

or

$$t = a\sqrt{\frac{2}{gh}}\int_0^x \frac{dx}{\sqrt{\left(1-x^2\right)\left(1-\frac{2a}{h}x^2\right)}},$$

by the transformations given in Art. 224.

As stated in Art. 225, either of these integrals is an Elliptic Integral of the First Class.

If $h < 2a$, the particle comes to rest when $y = h$. (See equation (2).) Denote this point on the circle by A. Denote the angle OCA by α.

Then $$h = a(1-\cos\alpha) = 2a\sin^2\tfrac{1}{2}\alpha.$$

Then, when $h < 2a$, $\frac{h}{2a} = \sin^2\frac{1}{2}\alpha$. Therefore in this case the value k in the elliptic integral is the sine of the half of the half angle of the whole swing of the particle.

EXERCISES

1. Show that the parabola of equation (6), Art. 281, has its vertex at

$$\left(\frac{v_0^2\sin 2\alpha}{2g}, \frac{v_0^2\sin^2\alpha}{2g}\right),$$

and its focus at $$\left(\frac{v_0^2\sin 2\alpha}{2g}, \frac{-v_0^2\cos 2\alpha}{2g}\right).$$

2. Find the horizontal range, and the time of flight for this range, of the projectile discussed in Art. 281.

Ans. Range $= \dfrac{v_0^2 \sin 2\alpha}{g}$. Time of flight $= \dfrac{2 v_0^2 \sin \alpha}{g}$.

3. Show that, for a maximum horizontal range, the initial direction of the path of projection must be 45°. Determine the position of the focus for this range.

Ans. The focus bisects the line that represents the horizontal range.

4. Find the range and time of flight of a projectile for a plane inclined β to the horizontal.

Ans. Range $= \dfrac{v_0^2}{g \cos^2 \beta}\Big[\sin(2\alpha - \beta) - \sin \beta\Big]$.

Time of flight $= \dfrac{2 v_0}{g \cos \beta} \sin(\alpha - \beta)$.

5. For given values of v_0 and β, find the value of α which makes the range of Exercise 4 a maximum.

Ans. $\alpha = \frac{1}{2}(90° + \beta)$.

6. In the cycloidal pendulum, find when the vertical velocity is greatest.

Ans. When one-half the vertical distance has been traversed.

7. What must be the length of the string in the cycloidal pendulum in order that it may beat seconds? *Ans.* 38.91 in.

8. What must be the length of the string in the simple pendulum in order that it may beat seconds?

Ans. 38.91 in. for small oscillations.

9. A simple pendulum swings through an angle of 180°. Find the time of oscillation.

Ans. $3.708\sqrt{\dfrac{a}{g}}$.

CHAPTER XXXIV

WORK AND ENERGY

286. **Work.** Work is said to be done on a body by a force when the point of application of the force has a component displacement in the direction of the force. Work is said to be done by a body against a force when the point of application of the force has a component displacement in a direction opposite to that of the force. In the latter case work may be said to be done on the body by the force if distinction be made between positive and negative work.

287. **Measurement of the work done on a particle in a given displacement when the force is constant in magnitude and direction, and motion is in a straight line.** If the force is constant in magnitude and direction, and motion takes place in a straight line, the work done by the force in a given displacement of the particle is measured by the product of the force and the component of the displacement along the line of force, the work being taken positive if the component of the displacement is in the direction of the force, and negative if it is in the direction opposite to that of the force.

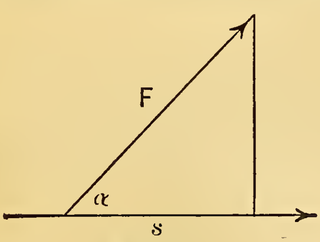


FIG. 118.

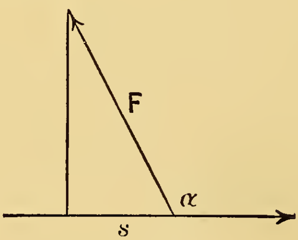


FIG. 119.

Thus, if w, F, s, and α (Figs. 118 and 119) represent the work done, the force, the displacement, and the angle between

the displacement and the direction of the force respectively, then

$$w \propto Fs \cos \alpha.$$

Since $w \propto Fs \cos \alpha$, $w = k\,Fs \cos \alpha$, where k is a constant depending on the units chosen for w, F, and s. As in Arts. 264 and 265, we shall find it convenient to choose k equal to 1. The unit of work will therefore be that work done when the body acted upon by the unit of force has a displacement of one unit in the direction of the force.

Since $F \cdot s \cos \alpha = F \cos \alpha \cdot s$, the work done is also measured by the product of the component of the force in the direction of the displacement and the displacement.

288. **Units of Work.** The systems of units of work in common use are the F.P.S. and C.G.S. gravitational systems, and the F.P.S. and C.G.S. absolute systems.

In the F.P.S. gravitational system, the unit of work is the work done when a force of one pound moves a body through a distance of one foot in its direction. This unit of work is called the **foot-pound.**

In the C.G.S. gravitational system, the unit of work is the work done when a force of one gram moves a body through a distance of one centimeter in its direction. This unit of work is called the **gram-centimeter.**

In the F.P.S. absolute system, the unit of work is the work done when a force of one poundal moves a body through a distance of one foot in its direction. This unit of work is called the **foot-poundal.**

In the C.G.S. absolute system, the unit of work is the work done when a force of one dyne moves a body through a distance of one centimeter in its direction. This unit of work is called the **erg.**

289. **Measurement of the work done on a particle in a given displacement when the force is variable in magnitude or direction or both.**

First, when motion is in a straight line.

Let AB (Fig. 120), where A and B have the abscissas a and b respectively, be the given displacement.

Divide AB into n equal parts. Call each part Δs.

Let F'_k and F''_k denote the least and greatest values respectively of the force that acts on the particle in an interval Δs of the displacement. Let ϕ'_k and ϕ''_k denote the greatest and least values respectively of the angle which the force makes with the displacement in this interval. Let w_k denote the work done by the force in the interval.

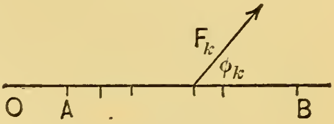


FIG. 120.

Then $$F'_k \cos \phi'_k \Delta s < w_k < F''_k \cos \phi''_k \Delta s. \tag{1}$$

Let F_k denote the force at the beginning of the interval Δs, and ϕ_k the angle which the force makes with the displacement at this point. Divide (1) by $F_k \cos \phi_k \Delta s$.

$$\therefore \frac{F'_k \cos \phi'_k}{F_k \cos \phi_k} < \frac{w_k}{F_k \cos \phi_k \Delta s} < \frac{F''_k \cos \phi''_k}{F_k \cos \phi_k}.$$

Now $$\underset{\Delta s \doteq 0}{\text{limit}} \left[\frac{F'_k \cos \phi'_k}{F_k \cos \phi_k} \right] = \underset{\Delta s \doteq 0}{\text{limit}} \left[\frac{F''_k \cos \phi''_k}{F_k \cos \phi_k} \right] = 1.$$

$$\therefore \underset{\Delta s \doteq 0}{\text{limit}} \left[\frac{w_k}{F_k \cos \phi_k \Delta s} \right] = 1.$$

Therefore, by the theorem of Art. 186, w_k may be replaced by $F_k \cos \phi_k \Delta s$ in any problem involving the limit of the sum of the infinitesimals w_k.

$$\therefore\ w = \underset{n=\infty}{\text{limit}} \sum_{x=a}^{x=b} F \cos \phi \Delta s$$

$$= \int_{x=a}^{x=b} F \cos \phi \, ds.$$

Second, when motion is along a curve.

It can be shown by a method which does not differ in im-

portant details from that given above, but is more complicated that the work done in this case is

$$\int_{x=a}^{x=b} F \cos \phi \, ds,$$

where F is the force at any point of the curve, ϕ the angle which the force makes with the tangent line to the curve at this point, and a and b the abscissas of the initial and final points respectively of the displacement.

We shall assume the theorem without discussion.

290. Work done by a resultant. If any number of forces act on a particle, the algebraic sum of the work done by the forces acting separately is the same as if the resultant of these forces alone acted on the particle.

Let R be the resultant of n forces $f_1, f_2, f_3, \cdots, f_n$, which act on a particle. Since R is the closing line of the polygon formed by adding the forces geometrically, its projection on any line is the sum of the projections of the forces $f_1, f_2, f_3, \cdots, f_n$. That is, if $R, f_1, f_2, f_3, \cdots, f_n$, make angles of $\theta, \theta_1, \theta_2, \theta_3, \cdots, \theta_n$ respectively with a line,

$$R \cos \theta = f_1 \cos \theta_1 + f_2 \cos \theta_2 + f_3 \cos \theta_3 + \cdots + f_n \cos \theta_n.$$

Let the line with which these forces make these angles be the tangent line to the path of displacement of the particle at any point. Therefore, by Art. 289, the work done by R between two points A and B on the path whose abscissas are a and b respectively is

$$\int_{x=a}^{x=b} R \cos \theta \, ds, \text{ or } \int_{x=a}^{x=b} (f_1 \cos \theta_1 + f_2 \cos \theta_2 + f_3 \cos \theta_3 + \cdots + f_n \cos \theta_n) \, ds.$$

But $\int_{x=a}^{x=b} (f_1 \cos \theta_1 + f_2 \cos \theta_2 + f_3 \cos \theta_3 + \cdots + f_n \cos \theta_n) \, ds$

$$= \int_{x=a}^{x=b} f_1 \cos \theta_1 \, ds + \int_{x=a}^{x=b} f_2 \cos \theta_2 \, ds + \int_{x=a}^{x=b} f_3 \cos \theta_3 \, ds + \cdots + \int_{x=a}^{x=b} f_n \cos \theta_n \, ds.$$

Therefore the algebraic sum of the work done by all the forces acting separately is the same as the work done if the resultant alone acted on the particle.

291. Central Force. A force which is always directed towards a fixed point and varies as the distance of its point of application from the fixed point varies is called a **central force.**

To prove that the work done by a central force is independent of the path.

Since a central force varies as the distance of its point of application from the fixed point varies, it is a function of its distance from the fixed point. Call it $f(r)$.

Take the fixed point as the pole and express the equation of the path in polar coördinates.

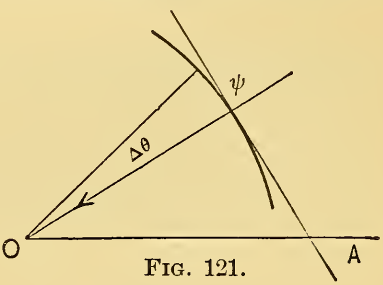


FIG. 121.

Then

$$\tan\psi = r\frac{d\theta}{dr}. \quad \text{(See Art. 102.)}$$

Since

$$\cos\psi = \frac{1}{\sec\psi} = \frac{1}{\sqrt{1+\tan^2\psi}}.$$

$$\therefore \cos\psi = \frac{1}{\sqrt{1+r^2\left(\dfrac{d\theta}{dr}\right)^2}} = \frac{dr}{\sqrt{dr^2+r^2d\theta^2}}.$$

$$\therefore \cos\psi = \frac{dr}{ds}. \quad \text{(See Art. 199.)}$$

$$\therefore dr = \cos\psi\, ds.$$

Therefore the work done

$$= \int f(r)\cos(\pi-\psi)\,ds = -\int f(r)\cos\psi\, ds,$$

$$= -\int f(r)\,dr, \tag{1}$$

the limits of integration in the last integral being the initial and final values of r.

Since (1) is a function of the initial and final values of r only, the work done is independent of the path.

292. Work expressed in terms of the components of the forces in the directions of the axes. Suppose that any number of forces act on a particle. Let X, Y, Z be the sum of the components of these forces in the directions of the axes. Let R be the resultant of the forces, and θ the angle which the tangent line to the path of displacement of the particle makes with R at any time. Let the direction cosines of the tangent line be $\cos\alpha$, $\cos\beta$, $\cos\gamma$. Then the work done by the given forces during a given displacement is

$$\int R\cos\theta\, ds, \text{ or} \int(X\cos\alpha + Y\cos\beta + Z\cos\gamma)ds.$$

Now $\qquad \cos\alpha\, ds = dx,\ \cos\beta\, ds = dy,\ \cos\gamma\, ds = dz.$

$$\therefore \int(X\cos\alpha + Y\cos\beta + Z\cos\gamma)\, ds = \int(X\,dx + Y\,dy + Z\,dz).$$

Therefore the work done by the forces during a given displacement of the particle is $\int(X\,dx + Y\,dy + Z\,dz)$ between proper limits.

293. Rate of Work. The **mean or average rate** at which work is done by a body is the work done in a period of time divided by the time.

The **rate of work** done by a body **at any particular instant** is the limit which the mean or average rate of work for a period of time immediately succeeding the instant in question approaches as the period of time is allowed to become indefinitely decreased.

294. Units of Rate of Work. The systems of units of rate of work derived from the units of work are the **foot-pound per second,** the **centimeter-gram per second,** the **foot-poundal per second,** and the **erg per second.**

The unit used in practice by American engineers is the **horse-power,** which is 550 foot-pounds per second. The French use

the unit 75 kilogrammeters per second, equivalent to 542486 foot-pounds per second, which they call the *force de cheval.*

A unit extensively used in electrical work is the **watt**, which is 10,000,000 ergs per second.

295. **Energy.** As stated in Art. 286, a body is said to do work against a force when the point of application of the force has a component displacement in a direction opposite to that of the force.

Definition. A body able to do work against a force is said to have work-power or **energy**.

296. **Units.** Energy, being power of doing work, is measured in terms of the units of work.

297. **Kinetic Energy.** If a body has a velocity, it is capable of doing work against a force that has a component opposite to the direction of the velocity.

Definition. The work-power or energy of a body due to a velocity is called the **kinetic energy** of the body.

298. **Determination of the kinetic energy expended in a given displacement of a particle.** Suppose that a particle of mass m has an initial velocity v_0, and is acted upon by a force F, which makes an angle θ with the opposite direction of the line of motion of the particle. To determine the kinetic energy expended in a given displacement of the particle.

Let the displacement of the particle be from A to B, where A and B have the abscissas a and b respectively (see Fig. 122).

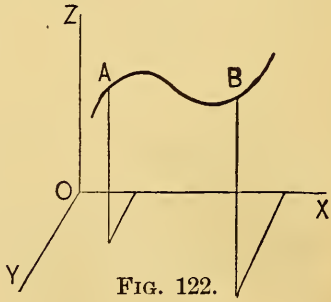


FIG. 122.

The equation of motion of the particle is

$$m\frac{d^2s}{dt^2} = -F\cos\theta,$$

or

$$mv\frac{dv}{ds} = -F\cos\theta, \text{ since } \frac{d^2s}{dt^2} = v\frac{dv}{ds}.$$

Integrate. $$\therefore \frac{mv^2}{2} = -\int F\cos\theta\, ds + c.$$

To determine c: Suppose for a moment that

$$\int F\cos\theta\, ds = \phi(x).$$

When $x = a$, $v = v_0$, and $\phi(x) = \phi(a)$. $\therefore c = \dfrac{mv_0^2}{2} + \phi(a)$.

$$\therefore \frac{mv^2}{2} - \frac{mv_0^2}{2} = -\phi(x) + \phi(a).$$

When $x = b$, $v =$ the velocity at B, and $\phi(x) = \phi(b)$. Denote the velocity at B by v_b.

$$\therefore \frac{mv_b^2}{2} - \frac{mv_0^2}{2} = -\left[\phi(b) - \phi(a)\right].$$

$$= -\int_{x=a}^{x=b} F\cos\theta\, ds.$$

$$\therefore \frac{mv_0^2}{2} - \frac{mv_b^2}{2} = \int_{x=a}^{x=b} F\cos\theta\, ds.$$

Now $\displaystyle\int_{x=a}^{x=b} F\cos\theta\, ds$ is the work done by the particle against F in its motion from A to B, and by definition this is the kinetic energy expended.

Therefore the kinetic energy expended is $\dfrac{mv_0^2}{2} - \dfrac{mv_b^2}{2}$.

When the particle comes to rest, $v_b = 0$. Therefore the kinetic energy of a particle $= \frac{1}{2}mv^2$, where v is the velocity of the particle.

299. **Potential Energy.** A body acted upon by a force directed towards a center, and in a position from which it can move towards the center, can, by virtue of its position, do work against another force having a component in a direction opposite to that of the first force. Thus, a weight in a position from which it can fall towards the center of the earth can do work against a force restraining it.

Definition. The work-power which a body possesses, due to its position, is called the energy of position, or the **potential energy** of the body.

300. A particle will possess potential energy in whatever position it may be placed in the field of force, but the nearer it comes to the center of force, the less potential energy it possesses. Suppose that a particle is acted upon by a force F, and is held from moving towards the center of force by a restraining force which has a component equal to F and in the opposite direction to it. If the attractive force F be increased by ever so small an amount, the work done in moving the particle from a point A to a point B at a distance s from A and in the line of force, differs by ever so small an amount from $\int F\,ds$, between proper limits. Then $\int F\,ds$, between proper limits, may be taken as the work done in moving the particle from A to B. Then the particle loses this amount of potential energy.

Similarly, if motion is in the opposite direction to that of the attractive force, the particle gains in potential energy by the amount of work done.

For example, it follows immediately from Art. 286 or 287 that the amount of work done in raising a weight of mass m to a height h is mgh. Since this amount of work is mgh, the potential energy gained by the weight by being raised a height h is therefore mgh. Similarly, if the weight be lowered through a distance h, the potential energy lost is mgh.

EXERCISES

1. Convert 1 foot-pound to gram-centimeters.

   Ans. 1 foot-pound $= 13{,}824.7$ gram-centimeters.

2. Convert 1 foot-poundal to ergs.

   Ans. 1 foot-poundal 421,358 ergs.

3. Convert 1 erg to gram-centimeters.

Ans. 1 erg $= 1019 \times 10^{-5}$ gram-centimeters.

4. A train weighs 100 tons. The resistance due to the air and friction is 8 pounds per ton. Find the work done in moving the train one mile up a grade of 1 in 400.

Ans. 6864000 foot-pounds.

5. A cylindrical cistern 30 feet deep and 6 feet in diameter is full of water. Find the work done in emptying the cistern. (The weight of a cubic foot of water $= 62\frac{1}{2}$ lb.)

Ans. 795200 foot-pounds (approx.).

6. Show that when a piston is driven by the pressure of an expanding fluid, the work done when the volume changes from v_0 to v_1 is $\int_{v_0}^{v_1} p\,dv$, where p is the pressure per unit area on the piston. Interpret this work as the area inclosed by the curve $p = f(v)$, the v-axis, and the ordinates corresponding to the abscissas v_0 and v_1 respectively.

7. When the volume of steam in the cylinder of the steam engine is 1.2 cubic feet, and the pressure is 60 pounds per square inch on the piston, the steam is cut off. Find the work done when the volume has changed to 3.2 cubic feet; given that the volume and pressure when the steam is cut off are connected by the equation $pv^{\frac{17}{16}} =$ a constant.

Ans. 9854 foot-pounds.

8. Steam is admitted into the cylinder of a steam engine under a constant pressure of 80 pounds per square inch. When the volume of steam in the cylinder increases from 0 to 2 cubic feet, the steam is cut off. The steam in the cylinder then expands under the law $pv^{\frac{17}{16}} =$ a constant until the volume is 6 cubic feet. It is then expelled under a constant back pressure of 10 pounds per square inch. Find the work done by the steam on the piston. *Ans.* 38860 foot-pounds (approx.).

9. Find the work done by an expanding fluid which obeys the law $pv^{\frac{4}{3}} = 11520$ when v changes from 0.5 cubic foot to 2 cubic feet. (p = number of pounds pressure per square foot; v = number of cubic feet.) *Ans.* 16112 foot-pounds.

10. Defining the average or mean value of $f(x)$ in the interval a to b to be the limit when $n = \infty$ of the average of n values of $f(x)$ corresponding to equidistant values of x in the interval a to b, show that the mean value of $f(x)$ in the interval a to b is

$$\frac{\int_a^b f(x)dx}{b-a}.$$

11. Find the mean pressure in Exercise 7.

Ans. 34.2 lb. per square inch.

12. Find the mean effective pressure of the steam in Exercise 8. *Ans.* 45 lb. per square inch.

13. A variable force has acted through 2 feet. The value of the force taken at seven equidistant points including the first and last is, in pounds, 200, 176, 142, 108, 76, 54, 20. Find, by Simpson's Rule, an approximation to the work done.

Ans. 223.1 foot-pounds.

14. Investigate Exercise 12, Chapter XXXII, for the work done until the spring is compressed to 6 inches.

Ans. $7\frac{1}{2}$ foot-pounds.

15. Investigate Exercise 13, Chapter XXXII, for the work done until the weight begins to descend. *Ans.* $\frac{1}{2}$ foot-pound.

16. Convert 1 horse-power to watts.

Ans. 1 horse-power = 742 watts.

17. Convert 1 *force de cheval* to ergs per second.

Ans. 1 *force de cheval* = 7.36×10^9 ergs per sec.

18. A mass of 20 pounds is being dragged along a horizontal plane, the rate of work done being $\frac{1}{10}$ horse-power. Find its acceleration when its speed is 2 miles per hour.

Ans. 30 ft.-per-sec. per sec. in the direction of its motion.

19. If the expansion in Exercise 9 above takes place in 0.1 second what is the horse-power? *Ans.* 292.9 horse-power.

20. A cannon ball weighing 200 pounds is discharged with a velocity of 1200 feet per second. Find its kinetic energy.
Ans. 45×10^5 foot-pounds.

21. In Exercise 13, Chapter XXXII, find the kinetic energy expended while the string contracts from 3 to 2 feet. *Ans.* 0.

22. A stone weighing 1 ton is lifted 30 feet. Find its increase in potential energy. *Ans.* 6×10^4 foot-pounds.

CHAPTER XXXV

ATTRACTION

301. **Definitions.** The **mean or average density** of a body is the mass of the body divided by the volume.

The **density of a body at a point** is the limit which the mean or average density of an element of volume containing the point approaches as the element of volume is allowed to become indefinitely decreased.

A body is said to be **homogeneous** when its density is the same at every point.

A body not homogeneous is said to be **heterogeneous.**

302. **Units of density.** The unit of density is unit mass divided by unit volume.

In the F.P.S. system there is probably no familiar substance whose density is the unit density. The mass of a cubic foot of water is approximately $62\frac{1}{2}$ times that of the ideal substance of unit density. Then in the F.P.S. system, the density of water is $62\frac{1}{2}$ times the unit density in this system.

The gram was chosen to be the mass of a cubic centimeter of distilled water at 4° C. Then in the C.G.S. system the unit of density is the density of distilled water at 4° C.

303. Every body in the universe attracts every other body with a force which depends in magnitude and direction on the masses and relative positions of the bodies.

In the case of two bodies considered as particles, the force of attraction varies directly as the product of the masses and inversely as the square of the distance between the particles. Thus, if F is the force of attraction between two particles of

masses m and m' respectively, and r is the distance between the particles,

$$F \propto \frac{mm'}{r^2}.$$

In the case of two continuous bodies, the force of attraction may be measured approximately by dividing the masses up into infinitesimal elements of mass, forming the expression $\frac{mm'}{r^2}$ for each pair of the elements, and taking the sum of the results. The actual force of attraction between the two bodies is the limit which this sum approaches as the number of elements is allowed to increase without limit while each element decreases without limit.

304. **Newton's Law of Gravitation.** The above law of gravitation, first enunciated by Newton in the case of particles, was derived by him from certain laws — called Kepler's laws — observed from the motions of planets, and may be accepted with as much confidence as was the fundamental law of motion, $F = ma$.

305. Since $F \propto \frac{mm'}{r^2}$, $F = \frac{kmm'}{r^2}$, where k is a constant depending on the units chosen for F, m, m', and r. In any system of units, k is evidently the force with which two particles each of unit mass attract each other when at unit distance apart.

Since $F = k\frac{mm'}{r^2}$ whatever be the masses of the attracting particles, the force of attraction between two particles, one of mass m, and the other of unit mass, is $\frac{km}{r^2}$.

Definition. The force of attraction between two particles, one of mass m and the other of unit mass, is called the force of attraction of the particle of mass m at the point where the unit particle is situated.

306. **Determination of k:** It will be shown later (see Art. 311) that the attraction of a homogeneous sphere on an external point is the same as if the mass of the sphere were con-

centrated at the center. Then, assuming this theorem, we have that, if m be the mass of the earth, assumed a homogeneous sphere, and m' the mass of the particle, the equation $F = k\frac{mm'}{r^2}$ expresses the force of attraction between the earth and particle. Divide the equation by m'. $\therefore \frac{F}{m'} = \frac{km}{r^2}$. Now $\frac{F}{m'}$ is the acceleration of the particle due to the attraction of the earth. Call it g, as in Art. 43. $\therefore g = \frac{km}{r^2}$. $\therefore k = \frac{gr^2}{m}$. The radius, r, and the density, ρ, of the earth are known to be 6.37×10^8 cm. and $5\frac{2}{3}$ respectively. In the C.G.S. absolute system of units, therefore,

$$k = \frac{3}{4}\,\frac{3200}{\pi \times 3.281 \times 5\frac{2}{3} \times 6.37 \times 10^8} = 645 \times 10^{-10}.$$

In this system of units, k is the force in dynes with which two particles each of mass one gram would attract each other when situated at a distance of one centimeter apart.

307. Astronomical System of Units. When the units are so selected that $k = 1$, the unit of mass is that mass which produces the unit of force in an equal mass when at unit distance from it. This system of units is called the **astronomical system** of units.

ATTRACTION AT A POINT DUE TO A SYSTEM OF PARTICLES

308. Let $m_1, m_2, m_3, \cdots, m_n$ be a system of n particles. To find the attraction at a point O due to these particles.

Let the origin of coördinates be taken at O. Let α_i, β_i, γ_i be the direction angles of the particles of mass m_i, $i = 1, 2, 3, \cdots, n$. Let X, Y, Z be the components of the resultant R in the directions of the axes.

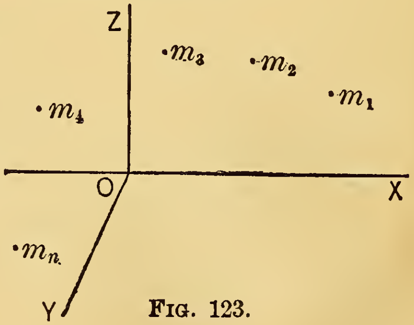


FIG. 123.

Then $$X = \frac{m_1 \cos \alpha_1}{r_1^2} + \frac{m_2 \cos \alpha_2}{r_2^2} + \frac{m_3 \cos \alpha_3}{r_3^2} + \cdots + \frac{m_n \cos \alpha_n}{r_n^2}$$

$$= \sum_{i=1}^{i=n} \frac{m_i \cos \alpha_i}{r_i^2}.$$

$$Y = \frac{m_1 \cos \beta_1}{r_1^2} + \frac{m_2 \cos \beta_2}{r_2^2} + \frac{m_3 \cos \beta_3}{r_3^2} + \cdots + \frac{m_n \cos \beta_n}{r_n^2}$$

$$= \sum_{i=1}^{i=n} \frac{m_i \cos \beta_i}{r_i^2}.$$

$$Z = \frac{m_1 \cos \gamma_1}{r_1^2} + \frac{m_2 \cos \gamma_2}{r_2^2} + \frac{m_3 \cos \gamma_3}{r_3^2} + \cdots + \frac{m_n \cos \gamma_n}{r_n^2}$$

$$= \sum_{i=1}^{i=n} \frac{m_i \cos \gamma_i}{r_i^2}.$$

The resultant attraction is $R = \sqrt{X^2 + Y^2 + Z^2}$, and the direction cosines of its line of application are $\frac{X}{R}$, $\frac{Y}{R}$, and $\frac{Z}{R}$ respectively.

ATTRACTION AT A POINT DUE TO A STRAIGHT HOMOGENEOUS WIRE

309. Let O be the given point, and AB the given wire. Choose the point O as the origin of coördinates, and the perpendicular from O on AB as the direction of the x-axis.

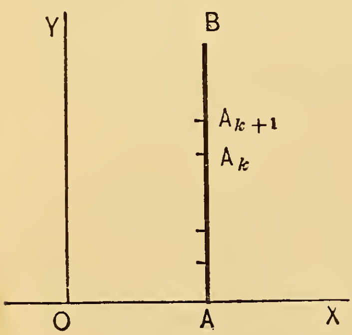


Fig. 124.

Let c denote the distance of O from AB, and b the length of AB. Let X and Y be the components of the attraction of AB in the directions of the axes. Denote the density of the wire by ρ.

We shall suppose here that the perpendicular from O on AB meets AB in A. The cases where it does not are left as exercises to the student.

Divide AB into n equal parts. Call each part Δy. Let A_k and A_{k+1}, with ordinates y_k and $y_k + \Delta y$ respectively, be two successive points of division of AB.

The resultant attraction at O, due to the element A_kA_{k+1}, is evidently greater than $\dfrac{k\rho\Delta y}{c^2 + (y_k + \Delta y)^2}$ and less than $\dfrac{k\rho\Delta y}{c^2 + y_k^2}$. The direction of the resultant attraction due to A_kA_{k+1} is along a line between OA_k and OA_{k+1}. Let α denote the angle which it makes with OA. Then the component of the attraction at O, due to the element A_kA_{k+1}, is greater than $\dfrac{k\rho\Delta y\cos\alpha}{c^2 + (y_k + \Delta y)^2}$ and less than $\dfrac{k\rho\Delta y\cos\alpha}{c^2 + y_k^2}$.

Let ϕ_k denote the angle AOA_k. Divide the inequality by $\dfrac{k\rho\Delta y\cos\phi_k}{c^2 + y_k^2}$, and pass to the limit. It is readily seen that the limit of each extreme is 1. Therefore each term in the component attraction of AB at O may be replaced by $\dfrac{k\rho\Delta y\cos\phi_k}{c^2 + y_k^2}$, or $\dfrac{k\rho c\Delta y}{(c^2 + y_k^2)^{\frac{3}{2}}}$, since $\cos\phi_k = \dfrac{c}{\sqrt{c^2 + y_k^2}}$.

$$\therefore\ X = \underset{n=\infty}{\text{limit}} \sum_{y=0}^{y=b} \frac{k\rho c\Delta y}{(c^2 + y^2)^{\frac{3}{2}}}$$

$$= ck\rho \int_0^b \frac{dy}{(c^2 + y^2)^{\frac{3}{2}}} = \frac{k\rho b}{c\sqrt{c^2 + b^2}} = \frac{k\rho}{c}\sin AOB.$$

$$\text{Similarly,}\quad Y = \underset{n=\infty}{\text{limit}} \sum_{y=0}^{y=b} \frac{k\rho y\Delta y}{(c^2 + y^2)^{\frac{3}{2}}}$$

$$= k\rho \int_0^b \frac{y\,dy}{(c^2 + y^2)^{\frac{3}{2}}}$$

$$= \frac{k\rho}{c}(1 - \cos AOB).$$

$$\text{Therefore}\quad R = \sqrt{X^2 + Y^2} = \frac{k\rho}{c}\sqrt{2(1 - \cos AOB)}$$

$$= \frac{2k\rho}{c}\sin\tfrac{1}{2}AOB,$$

and its line of action makes with OA an angle whose tangent is

$$\frac{Y}{X}=\frac{1-\sin OBA}{\cos OBA}=\frac{1-\cos AOB}{\sin AOB}=\tan\tfrac{1}{2}AOB.$$

The resultant attraction of AB at O therefore bisects the angle AOB.

ATTRACTION OF A HOMOGENEOUS CIRCULAR DISC AT A POINT IN ITS AXIS

310. Let O be the point, and ABC the given circular disc. Choose the center of the disc as the pole, and a line PA in the disc as the initial line.

Let c denote the distance of O from the pole, and a the radius of the disc. Let X and Y denote the components of the attraction of the disc in the direction of the axis, and perpendicular to the axis respectively. Let ρ denote the density of the disc.

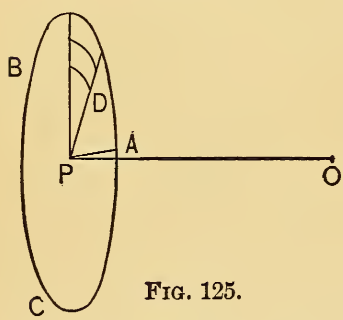


FIG. 125.

Divide the disc up into polar elements of area in the same manner as if an area were sought by double integration. Let (r_ι, θ_k) be the coördinates of the corner D of the element indicated. (Fig. 125.)

The resultant attraction at O due to the element indicated is greater than

$$\frac{k\rho(r_\iota\Delta r\Delta\theta+\frac{1}{2}\Delta r^2\Delta\theta)}{c^2+(r_\iota+\Delta r)^2}$$

and less than

$$\frac{k\rho(r_\iota\Delta r\Delta\theta+\frac{1}{2}\Delta r^2\Delta\theta)}{c^2+r_\iota^2}.$$

Let the resultant attraction of this element make an angle α with PO. Then the component of the attraction of this element in the direction of the axis is greater than

$$\frac{k\rho(r_\iota\Delta r\Delta\theta+\frac{1}{2}\Delta r^2\Delta\theta)\cos\alpha}{c^2+(r_\iota+\Delta r)^2}$$

and less than $\dfrac{k\rho(r_i\Delta r\Delta\theta + \frac{1}{2}\Delta r^2\Delta\theta)\cos\alpha}{c^2 + r_i^2}$. Let ϕ_k denote the angle DOP. Divide the inequality by $\dfrac{k\rho r_i\Delta r\Delta\theta\cos\phi_k}{c^2 + r_i^2}$ and pass to limits. The limit of each extreme, as Δr and $\Delta\theta$ both approach zero, is 1. Therefore each term in the component attraction of the disc in the direction of the axis may be replaced in the double limit by $\dfrac{k\rho r_i\Delta r\Delta\theta\cos\phi_k}{c^2 + r_i^2}$, or by $\dfrac{k\rho c r_i\Delta r\Delta\theta}{(c^2 + r_i^2)^{\frac{3}{2}}}$, since $\cos\phi_k = \dfrac{c}{\sqrt{c^2 + r_i^2}}$.

Therefore
$$X = k\rho c\int_0^a\int_0^{2\pi}\frac{r\,dr\,d\theta}{(c^2 + r^2)^{\frac{3}{2}}}$$
$$= 2\pi k\rho\left[1 - \frac{c}{\sqrt{c^2 + r^2}}\right].$$

From the symmetry of the figure, or by direct calculation, it may be seen that Y is zero. The resultant attraction of the disc at O is therefore $2\pi k\rho\left[1 - \dfrac{c}{\sqrt{c^2 + r^2}}\right]$, and its line of action is along PO.

ATTRACTION OF A HOMOGENEOUS SPHERICAL SHELL

311. Let r_0 be the inner and r_1 the outer radius of the shell. Let P be the center of the shell and O, distant c from P, the point of attraction. Through OP pass a plane OPH. Divide the area of the circle formed by the intersection of the sphere and the plane up into infinitesimal elements in the same manner as if an area were sought by double integration. Let (r_i, θ_k) be the coördinates of a corner of the element of area indicated.

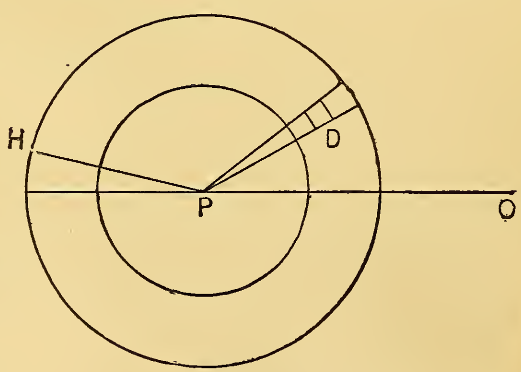


FIG. 126.

If the element of area indicated be revolved about the line OP, the volume generated is such that the limit of its ratio to $2\pi r_i^2 \sin\theta_k \Delta r \Delta\theta$ as Δr and $\Delta\theta$ both approach zero is 1. (See Art. 217.) Revolve the element of area through an angle $\Delta\phi$. The volume generated is to $2\pi r_i^2 \sin\theta_k \Delta r\,\Delta\theta$ as $\Delta\phi$ is to 2π. The volume generated by revolving the area through an angle $\Delta\phi$ may then be taken in the limit, as Δr and $\Delta\theta$ both approach zero, as $r_i^2 \sin\theta_k \Delta r \Delta\theta \Delta\phi$.

The attraction at O of the mass whose volume is $r_i^2 \sin\theta_k \Delta r \Delta\theta \Delta\phi$ is, on the supposition that the whole mass is concentrated at D, $\dfrac{k\rho r_i^2 \sin\theta_k \Delta r \Delta\theta \Delta\phi}{y^2}$ where $y^2 = \overline{OD}^2 = c^2 + r^2 - 2cr\cos\theta_k$. The component of this attraction in the direction of PO is therefore

$$\frac{k\rho r_i^2 \sin\theta_k \Delta r \Delta\theta \Delta\phi \cos OPD}{y^2},$$

or
$$\frac{k\rho r_i^2 \sin\theta_k (c - r_i\cos\theta_k)\Delta r \Delta\theta \Delta\phi}{y^3}, \tag{1}$$

since
$$\cos OPD = \frac{c - r_i\cos\theta_k}{y}.$$

It is evident from symmetry that the component of this attraction in the directions PY and PZ, perpendicular to PO and each other, are zero. The resultant attraction at O due to a mass whose volume is $r_i^2 \sin\theta_k \Delta r \Delta\theta \Delta\phi$, situated at D, is therefore (1), and its line of action is along PO.

It can be shown by an investigation, presenting no difficulty other than that of length, that the attraction at O, due to the element of mass whose volume is indicated in the figure, lies between two extremes, one of which is (1), and that the limit of the ratio of either extreme to this attraction is 1. Assuming that this limit is 1, we have, from the theorem of Art. 186, that the attraction of the element of mass may be taken as (1) in the limit.

The attraction at O of the whole spherical shell is therefore

$$k\rho \int_{r_0}^{r_1}\int_0^{\pi}\int_0^{2\pi} \frac{r^2(c - r\cos\theta)\sin\theta\, dr\, d\theta\, d\phi}{y^3}.$$

Substitute for θ its value in terms of y. Therefore the attraction

$$=\frac{k\rho}{2\,c^2}\int_{r_0}^{r_1}\int_{y_0}^{y_1}\int_0^{2\pi}\frac{r(c^2-r^2+y^2)}{y^2}\,dr\,dy\,d\phi$$

$$=\frac{\pi k\rho}{c^2}\int_{r_0}^{r_1} r\left[\frac{r^2-c^2+y^2}{y}\right]\Bigg|_{y_0}^{y_1} dr.$$

To determine y_0 and y_1, we must distinguish between two cases.

Case I. Where O is a point inside the shell.

Then $\quad y_0=r-c,$ and $y_1=r+c.$

Therefore the attraction $=\dfrac{\pi k\rho}{c^2}\displaystyle\int_{r_0}^{r_1} 0\,dr=0.$

Case II. Where O is a point outside the shell.

Then $\quad y_0=c-r,$ and $y_1=c+r.$

Therefore the attraction $=\dfrac{4\,\pi k\rho}{c^2}\displaystyle\int_{r_0}^{r_1} r^2 dr$

$$=\frac{4\,\pi k\rho}{3\,c^2}\left(r_1^{\,3}-r_0^{\,3}\right). \tag{2}$$

If in (2) we make $r_0=0$, we have the attraction of a solid sphere of radius r_1 and density ρ at a point outside the sphere and distant c from the center to be

$$\frac{4\,\pi k\rho r_1^{\,3}}{3\,c^2}.$$

EXERCISES

1. Show that the value of k in the F.P.S. absolute system equals 1033×10^{-12}. (See Art. 306.)

2. Find the attraction due to a homogeneous straight wire of length $2\,l$ at a point in the line of the wire and distant c from one end. *Ans.* $\dfrac{2\,k\rho l}{c(c+2\,l)}$, in the direction of the wire.

3. Find the attraction due to a homogeneous straight wire of length $2l$ at a point distant c from the center of the wire and on a line perpendicular to the wire.

Ans. $\dfrac{2k\rho l}{c\sqrt{c^2+l^2}}$, in the direction of the line perpendicular to the wire.

4. Find the attraction due to a homogeneous right circular cylinder of length $2l$ and radius of cross section a, at a point in the axis produced of the cylinder and distant c from one end.

Ans. $2\pi k\rho[2l+\sqrt{a^2+c^2}-\sqrt{(c+2l)^2+a^2}]$, in the direction of the axis of the cylinder.

5. From the result in Exercise 4, show that the attraction of a homogeneous right circular cylinder of radius of cross section a, and infinite in one direction, at the point of intersection of the axis and base, is $2\pi k\rho a$.

6. Three rods, equal in length and of uniform density, form a triangle. Find the point at which the resultant attraction is zero. *Ans.* The center of the inscribed circle.

7. Find the attraction of a homogeneous straight wire of infinite length at a point c units distant from it.

Ans. $\dfrac{2k\rho}{c}$, in the direction perpendicular to the wire.

8. Find the attraction of a homogeneous right circular cone of vertical angle 2α and height h, at the vertex of the cone.

Ans. $2\pi k\rho(1-\cos\alpha)h$, in the direction of the axis.

9. Find the attraction of a homogeneous right circular cylinder of iron, 10 meters long, radius of cross section 1 meter, upon a mass of 1 kilogram in the axis of the cylinder, distant 10 cm. from one end. (Weight of iron = 7.23 grams per cu. cm.)

Ans. 0.000256 gram, in the direction of the axis of the cylinder.

10. Supposing the earth at rest and the resistance of the air zero, with what velocity must a body be projected vertically upwards from the surface of the earth in order that it may never return? (See Art. 278. Assume radius of earth = 4000 miles.) *Ans.* 36800 ft. per sec. (nearly).

CHAPTER XXXVI

CENTERS OF GRAVITY

Definition. Two parallel forces are said to be **like** or **unlike** according as they act in the same direction or in opposite directions.

312. Suppose that there are two like parallel forces acting at points in a rigid body. To determine the resultant of the forces.

Let the forces be P_1 and P_2, acting at the points A and B respectively (see Fig. 127).

At A introduce a force F acting in the direction BA, and at B an equal force F acting in the direction AB. Complete the parallelograms AC and BD.

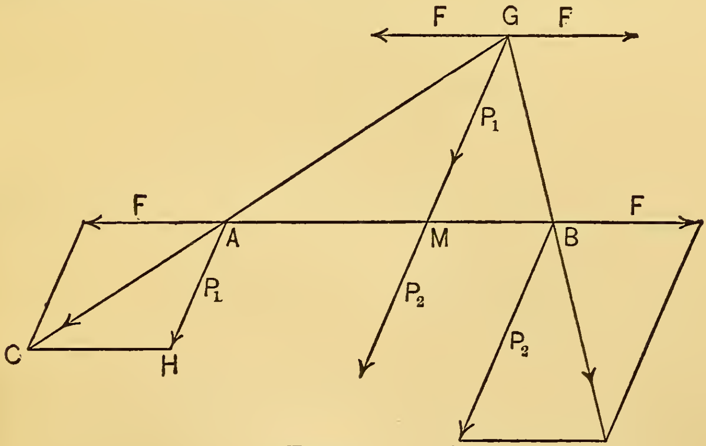


FIG. 127.

The forces F and P_1 at A, and the forces F and P_2 at B are equivalent to two forces represented in magnitude and direction by AC and BD respectively, and, since the forces F neutralize

each other, the forces P_1 and P_2 are therefore equivalent to these forces.

Produce CA and DB to meet in G. At G resolve the forces represented by AC and BD into forces equal and parallel to the original ones, F, P_1, and P_2.

Since the forces F at G neutralize each other, the resultant of the forces acting at A and B is equivalent to the force $P_1 + P_2$ acting at G.

To determine the point at which the line of action of the resultant force cuts the line AB:

The triangles GMA and AHC are similar.

$$\therefore \frac{GM}{MA} = \frac{AH}{HC} = \frac{P_1}{F}.$$

The triangles GMB and BKD are similar.

$$\therefore \frac{GM}{MB} = \frac{BK}{KD} = \frac{P_2}{-F}.$$

Therefore, by division, $$\frac{MB}{MA} = -\frac{P_1}{P_2}.$$

$$\therefore -P_1 \cdot MA = P_2 \cdot MB.$$

$$\therefore P_1 \cdot AM = P_2 \cdot MB.$$

The resultant of two like parallel forces P_1 and P_2, acting at points A and B in a rigid body, is therefore a force $P_1 + P_2$ parallel to P_1 and P_2, acting at a point M on the line joining A and B, such that

$$P_1 \cdot AM = P_2 \cdot MB.$$

Since $P_1 \cdot AM = P_2 \cdot MB$, whatever be the direction of the forces P_1 and P_2, it follows that the same point M would be found if P_1 and P_2 were revolved about their points of application, remaining parallel.

313. Given the coördinates of the points A and B at which the forces P_1 and P_2 act, we can find the coördinates of the point M, through which their resultant acts, as follows:

Let the coördinates of A, B, and M be (x_1, y_1, z_1), (x_2, y_2, z_2), and $(\bar{x}, \bar{y}, \bar{z})$ respectively.

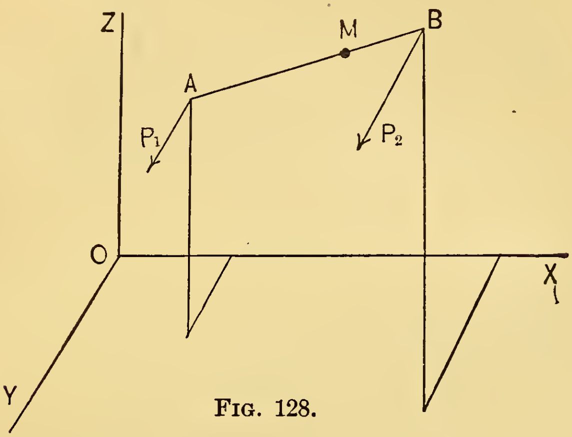


FIG. 128.

Since $P_1 \cdot AM = P_2 \cdot MB$,

$$\therefore \; P_1(\bar{x} - x_1) = P_2(x_2 - \bar{x}).$$

$$\therefore \; (P_1 + P_2)\bar{x} = P_1x_1 + P_2x_2.$$

$$\therefore \; \bar{x} = \frac{P_1x_1 + P_2x_2}{P_1 + P_2}.$$

Similarly,

$$\bar{y} = \frac{P_1y_1 + P_2y_2}{P_1 + P_2},$$

and

$$z = \frac{P_1z_1 + P_2z_2}{P_1 + P_2}.$$

314. Suppose that there are n like parallel forces, P_1, P_2, P_3, $\cdots$, P_n, acting at points A_1, A_2, A_3, $\cdots$, A_n respectively, in a rigid body. To determine the resultant of the forces.

Suppose that the points of application of the forces are given by their coördinates. Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3),

$\cdots$, (x_n, y_n, z_n) be the coördinates of the points A_1, A_2, A_3, $\cdots$, A_n respectively.

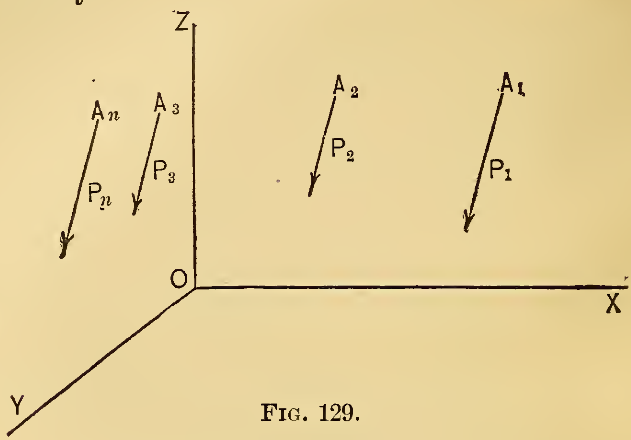


FIG. 129.

By Art. 313, the resultant of P_1 and P_2 is a force $P_1 + P_2$ acting at a point M_1, coördinates (x', y', z'), such that

$$P_1(x' - x_1) = P_2(x_2 - x'),$$

$$P_1(y' - y_1) = P_2(y_2 - y'),$$

$$P_1(z' - z_1) = P_2(z_2 - z').$$

$$\therefore\ x' = \frac{P_1x_1 + P_2x_2}{P_1 + P_2},\quad y' = \frac{P_1y_1 + P_2y_2}{P_1 + P_2},\quad z' = \frac{P_1z_1 + P_2z_2}{P_1 + P_2}.$$

Similarly, the resultant of $P_1 + P_2$, acting at M_1, and P_3 at A_3, is a force $P_1 + P_2 + P_3$, acting at a point M_2, coördinates (x'', y'', z''), such that

$$(P_1 + P_2)(x'' - x') = P_3(x_3 - x''),$$

$$(P_1 + P_2)(y'' - y') = P_3(y_3 - y''),$$

$$(P_1 + P_2)(z'' - z') = P_3(z_3 - z'').$$

$$\therefore\ x'' = \frac{(P_1 + P_2)x' + P_3x_3}{P_1 + P_2 + P_3} = \frac{P_1x_1 + P_2x_2 + P_3x_3}{P_1 + P_2 + P_3},$$

$$y'' = \frac{(P_1 + P_2)y' + P_3y_3}{P_1 + P_2 + P_3} = \frac{P_1y_1 + P_2y_2 + P_3y_3}{P_1 + P_2 + P_3},$$

$$z'' = \frac{(P_1 + P_2)z' + P_3z_3}{P_1 + P_2 + P_3} = \frac{P_1z_1 + P_2z_2 + P_3z_3}{P_1 + P_2 + P_3}.$$

By continuing this process, we find finally that the resultant of the n like parallel forces is a force $P_1 + P_2 + P_3 + \cdots + P_n$, acting at a point M_{n-1}, coördinates $(\bar{x}, \bar{y}, \bar{z})$, such that

$$\bar{x} = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \cdots + P_nx_n}{P_1 + P_2 + P_3 + \cdots + P_n},$$

$$\bar{y} = \frac{P_1y_1 + P_2y_2 + P_3y_3 + \cdots + P_ny_n}{P_1 + P_2 + P_3 + \cdots + P_n},$$

$$\bar{z} = \frac{P_1z_1 + P_2z_2 + P_3z_3 + \cdots + P_nz_n}{P_1 + P_2 + P_3 + \cdots + P_n}.$$

Since the expressions for $\bar{x}$, $\bar{y}$, and $\bar{z}$ are independent of the direction of the forces, it follows that the same point M_{n-1} will be found if the forces be revolved about their points of application remaining parallel. Also, since a variation of the order in which the forces were combined would cause only a variation in the order of the terms in $\bar{x}$, $\bar{y}$, and $\bar{z}$, it follows that the same point M_{n-1} will be determined if the forces be combined in any other order.

Definition. The point M_{n-1}, coördinates $(\bar{x}, \bar{y}, \bar{z})$, is called the center of the system of parallel forces.

315. An important special case of like parallel forces acting at points in a rigid body is that of gravity acting on n particles rigidly connected.

Let $m_1, m_2, m_3, \cdots, m_n$ be the masses of the particles. Then, by the preceding article, the coördinates of the center of the system of these parallel forces acting on these masses are

$$\bar{x} = \frac{m_1gx_1 + m_2gx_2 + m_3gx_3 + \cdots + m_ngx_n}{m_1g + m_2g + m_3g + \cdots + m_ng}$$

$$= \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n}{m_1 + m_2 + m_3 + \cdots m_n}.$$

$$\bar{y} = \frac{m_1gy_1 + m_2gy_2 + m_3gy_3 + \cdots + m_ngy_n}{m_1g + m_2g + m_3g + \cdots + m_ng}$$

$$= \frac{m_1y_1 + m_2y_2 + m_3y_3 + \cdots + m_ny_n}{m_1 + m_2 + m_3 + \cdots + m_n}.$$

$$\bar{z} = \frac{m_1gz_1 + m_2gz_2 + m_3gz_3 + \cdots + m_ngz_n}{m_1g + m_2g + m_3g + \cdots + m_ng}$$

$$= \frac{m_1z_1 + m_2z_2 + m_3z_3 + \cdots + m_nz_n}{m_1 + m_2 + m_3 + \cdots + m_n}.$$

Definition. The center of the system of parallel forces of gravity acting on a system of particles rigidly connected is called the **center of gravity** of the system of particles.

Center of gravity is usually denoted by the letters C.G.

316. Let a force P have its line of action FB oblique to the line AC (see Fig. 130).

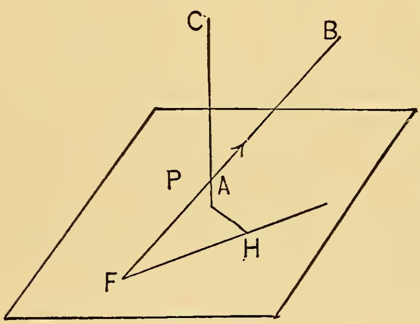


FIG. 130.

At A draw a plane perpendicular to AC, cutting FB in F. Let FH be the projection of FB on this plane. From A draw a perpendicular to FH.

Denote the angle HFB by ϕ, and the length of the perpendicular from A to FH by a.

Definition of moment. The moment of the force P about the line AC is defined to be $aP\cos\phi$.

317. From the definition of moment, it follows that:

First: If the line of action of P is perpendicular to AC, then $\phi = 0$, and therefore the moment of P about AC is the product of the magnitude of P and the perpendicular distance of the line of action of P from AC.

Second: If AC and the line of action of P are in the same plane and not parallel, a is zero, and therefore the moment of P about AC is zero.

If AC and the line of action of P are parallel, $\phi = 90°$, and therefore the moment of P about AC is zero.

318. The equations of Art. 315 may be written in the form

$$\bar{x} \cdot \Sigma m = \Sigma mx,$$

$$\bar{y} \cdot \Sigma m = \Sigma my,$$

$$\bar{z} \cdot \Sigma m = \Sigma mz.$$

From these equations we see that the moment of the sum of the masses, considered as concentrated at the C.G., about any one of the coördinate axes, is equal to the sum of the moments of the masses about that axis.

319. In the case of a continuous body bounded by known surfaces, the coördinates of the C.G., $(\bar{x}, \bar{y}, \bar{z})$, may be found approximately by dividing the body up into infinitesimal elements of mass Δm and determining $\bar{x}$, $\bar{y}$, and $\bar{z}$ so that

$$\bar{x} = \frac{\Sigma x \, \Delta m}{\Sigma \, \Delta m},$$

$$\bar{y} = \frac{\Sigma y \, \Delta m}{\Sigma \, \Delta m},$$

$$\bar{z} = \frac{\Sigma z \, \Delta m}{\Sigma \, \Delta m},$$

where the summation includes all the elements of mass, Δm, and x, y, and z are the distances from the axes of some point in Δm.

Definition. The limits which $\bar{x}$, y, and $\bar{z}$ approach as Δm approaches zero determine the coördinates of the C.G. of the body. Thus,

$$\bar{x} = \frac{\int x \, dm}{\int dm}, \quad \bar{y} = \frac{\int y \, dm}{\int dm}, \quad \bar{z} = \frac{\int z \, dm}{\int dm},$$

where the limits of integration are such as to embrace the whole solid.

320. The above equations may be written

$$\bar{x}\int dm = \int x\,dm,$$
$$\bar{y}\int dm = \int y\,dm,$$
$$\bar{z}\int dm = \int z\,dm.$$

From these equations we see that the moment of the body considered as concentrated at C.G., about any one of the coördinate axes, is equal to the limit of the sum of the moments of the elements about that axis.

321. Find the coördinates of the C.G. of a homogeneous plane area when the equation of the bounding curve is given in rectangular coördinates.

Suppose that the bounding curve is as in Fig. 131.

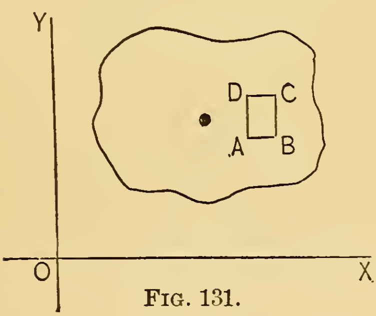


FIG. 131.

Divide the area up into rectangular elements of area $\Delta x\,\Delta y$ in the same manner as if an area by double integration were sought.

Let $(\bar{x}, \bar{y})$ be the coördinates of the C.G. The mass of each element is $\rho\,\Delta x\,\Delta y$, where ρ is the mass per unit of area. Take an element $ABCD$. Suppose that the coördinates of A are (x_k, y_l). Then the coördinates of B are $(x_k + \Delta x, y_l)$.

The moment of the element $ABCD$ about the y-axis is evidently greater than if the whole mass were concentrated at A and less than if it were concentrated at B.

$$\therefore\ x_k\rho\,\Delta x\,\Delta y < \begin{matrix}\text{moment of } ABCD\\ \text{about the } y\text{-axis}\end{matrix} < (x_k + \Delta x)\rho\,\Delta x\,\Delta y.$$

Divide the inequality by $x_k\rho\,\Delta x\,\Delta y$ and pass to the limit. Since the limit of each extreme is 1,

$$\therefore\ \underset{n=\infty}{\text{limit}}\left[\frac{\text{moment of } ABCD \text{ about the } y\text{-axis}}{x_k\rho\,\Delta x\,\Delta y}\right] = 1.$$

$$\therefore \bar{x} = \frac{\iint \rho x \, dx \, dy}{\iint \rho \, dx \, dy}.$$

Similarly, $$y = \frac{\iint \rho y \, dx \, dy}{\iint \rho \, dx \, dy}.$$

In these integrals the limits are the same as if an area were sought.

Since ρ is constant it may be removed from under the integral signs and cancelled.

Therefore $$\bar{x} = \frac{\iint x \, dx \, dy}{\iint dx \, dy}, \text{ and } \bar{y} = \frac{\iint y \, dx \, dy}{\iint dx \, dy}.$$

Since $\iint dx \, dy$ denotes the area, we have

$$\bar{x} = \frac{\iint x \, dx \, dy}{\text{area}}, \text{ and } \bar{y} = \frac{\iint y \, dx \, dy}{\text{area}}.$$

322. Find the coördinates of the C.G. of a homogeneous plane area when the equation of the bounding curve is given in polar coördinates.

Suppose that the curve is as in Fig. 132.

Divide the area up into polar elements of area

$$r\Delta\theta \, \Delta r + \tfrac{1}{2} \Delta r^2 \, \Delta\theta$$

in the same manner as if an area were sought.

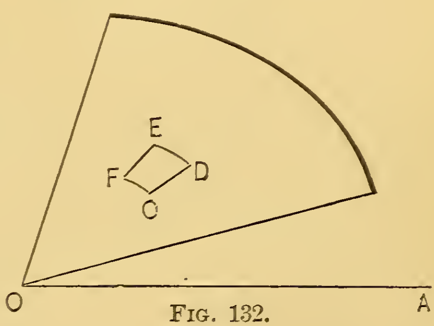


FIG. 132.

Let the coördinates of the C.G. be $(\bar{x}, \bar{y})$, where $\bar{x}$ is the distance of the point from a line OB perpendicular to the initial line OA, and $\bar{y}$ is the distance of the point from OA.

The mass of each element is $\rho(r\Delta\theta \, \Delta r + \frac{1}{2} \Delta r^2 \, \Delta\theta)$.

Take an element $CDEF$. Suppose that the coördinates of C are (r_l, θ_k). Then the coördinates of D are $(r_l + \Delta r, \theta_k)$ and the coördinates of F are $(r_l, \theta_k + \Delta\theta)$.

The moment of $CDEF$ about OB is evidently greater than if the whole mass were concentrated at F and less than if it were concentrated at D.

$$\therefore\ r_l\rho\cos(\theta_k + \Delta\theta)(r_l\,\Delta\theta\,\Delta r + \tfrac{1}{2}\,\Delta r^2\,\Delta\theta) < \text{moment of } CDEF$$
$$\text{about } OB < (r_l + \Delta r)\rho\cos\theta_k(r_l\,\Delta\theta\,\Delta r + \tfrac{1}{2}\,\Delta r^2\,\Delta\theta).$$

Divide the inequality by $r_l\rho\cos\theta_k(r_l\,\Delta\theta\,\Delta r)$ or $r_l^2\rho\cos\theta_k\,\Delta\theta\,\Delta r$, and pass to the limit. Since the limit of each extreme is 1,

$$\therefore\ \lim_{n=\infty}\left[\frac{\text{moment of } CDEF \text{ about } OB}{r_l^2\rho\cos\theta_k\,\Delta\theta\,\Delta r}\right] = 1.$$

$$\therefore\ \bar{x} = \frac{\displaystyle\iint r^2\rho\cos\theta\,d\theta\,dr}{\displaystyle\iint r\rho\,d\theta\,dr}.$$

Similarly, $$\bar{y} = \frac{\displaystyle\iint r^2\rho\sin\theta\,d\theta\,dr}{\displaystyle\iint r\rho\,d\theta\,dr}.$$

Or, as in the preceding article,

$$\bar{x} = \frac{\displaystyle\iint r^2\cos\theta\,d\theta\,dr}{\text{area}},\ \text{and}\ \bar{y} = \frac{\displaystyle\iint r^2\sin\theta\,d\theta\,dr}{\text{area}}.$$

323. By methods entirely similar to those of Arts. 321 and 322, we can find that the coördinates of the C.G.:

Of a homogeneous arc when the equation of the bounding curve is given in rectangular coördinates are:

$$\bar{x} = \frac{\displaystyle\int x\,ds}{\displaystyle\int ds},$$

$$\bar{y} = \frac{\displaystyle\int y\,ds}{\displaystyle\int ds}.$$

Of a homogeneous arc when the equation of the bounding curve is given in polar coördinates are:

$$\bar{x} = \frac{\int r\cos\theta\,ds}{\int ds}, \qquad \bar{y} = \frac{\int r\sin\theta\,ds}{\int ds}.$$

Of a homogeneous volume of revolution when the equation of the generating arc is given in rectangular coördinates are:

$$\bar{x} = \frac{\int\int xy\,dx\,dy}{\int\int y\,dx\,dy},$$

$$\bar{y} = 0.$$

Of a homogeneous volume of revolution when the equation of the generating arc is given in polar coördinates are:

$$\bar{x} = \frac{\int\int r^3 \sin\theta\cos\theta\,d\theta\,dr}{\int\int r^2 \sin\theta\,d\theta\,dr},$$

$$\bar{y} = 0.$$

Of a homogeneous surface of revolution when the equation of the generating arc is given in rectangular coördinates are:

$$\bar{x} = \frac{\int xy\,ds}{\int y\,ds},$$

$$\bar{y} = 0.$$

324. To find the coördinates of the C.G. of a given homogeneous plane triangle.

Let the triangle and its coördinate axes be as in Fig. 133.

Let $(\bar{x}, \bar{y})$ be the coördinates of the C.G.

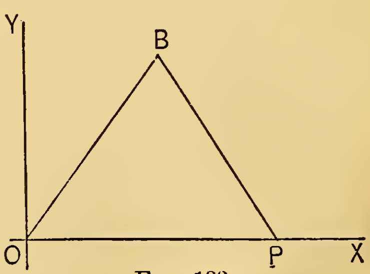


Fig. 133.

The equations of the lines OB and PB are

$$y = \frac{c}{b}x, \text{ and } y = \frac{c}{a-b}(a-x) \text{ respectively.}$$

$$\therefore\ \bar{x} = \frac{\int_0^b \int_0^{\frac{cx}{b}} x\,dx\,dy + \int_b^a \int_0^{\frac{c}{a-b}(a-x)} x\,dx\,dy}{\text{area of } \Delta} = \frac{a+b}{3},$$

and

$$\bar{y} = \frac{\int_0^b \int_0^{\frac{cx}{b}} y\,dx\,dy + \int_b^a \int_0^{\frac{c}{a-b}(a-x)} y\,dx\,dy}{\text{area of } \Delta} = \frac{c}{3}.$$

It will be remembered from analytic geometry that $\left(\frac{a+b}{3}, \frac{c}{3}\right)$ are the coördinates of the point of intersection of the medians of the triangle. The C.G. of a plane triangle is therefore the point of intersection of the medians.

325. If a given area is the sum or the difference of the areas of figures whose areas and centers of gravity are known, the C.G. of the given area can be found without resort to integration.

Example. The side OG of the homogeneous square $OBDG$ (Fig. 134) is bisected in A, and the triangle OBA is turned about AB to the position OBC. To find the C.G. of the shaded area.

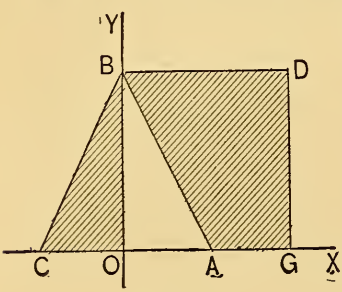


Fig. 134.

Let the sides of the square be $2a$. Take the coördinate axes as indicated.

The C.G. of the square is evidently at the point whose coördinates are (a, a). Let $(\bar{x}, \bar{y})$ be the coördinates of the C.G. of the shaded area. The area of the shaded figure $= 4a^2$. The areas of the triangles OBC and OBA are each $= a^2$. The C.G. of OBA is $\left(\frac{a}{3}, \frac{2a}{3}\right)$, and of OBC is $\left(-\frac{a}{3}, \frac{2a}{3}\right)$. (See Art. 324.)

The moment of the shaded figure about the y-axis = moment of square, *minus* moment of triangle OBA, *minus* moment of triangle OBC, each about the y-axis.

$$\therefore\ 4a^2\cdot\bar{x}=4a^2\cdot a-a^2\cdot\frac{a}{3}-a^2\cdot\frac{a}{3}.$$

$$\therefore\ \bar{x}=\tfrac{5}{6}a.$$

Similarly, $\bar{y}=a$, a result which would be expected from the symmetry of the figure.

THEOREMS OF PAPPUS

326. Theorem I. The area of the surface generated by the revolution of an arc of a plane curve about an axis in its plane and not crossing the arc is equal to the product of the length of the arc and the length of the path described by the C.G. of the arc.

Proof. Let S denote the area of the surface, s the length of the arc, and $\bar{y}$ the distance of the C.G. of the arc from the line.

Then $S=2\pi\int y\,ds$. (See Art. 200.) Also, $\bar{y}=\dfrac{\int y\,ds}{s}$. (See Art. 323.)

$$\therefore\ S=s\cdot 2\pi\bar{y}.$$

Theorem II. The volume generated by the revolution of a plane area about an axis in its plane and not crossing the area is equal to the product of the area and the length of the path described by the C.G. of the area.

Proof. Let V denote the volume, A the area, and $\bar{y}$ the distance of the C.G. of the area from the line.

Then $V=\pi\int y^2\,dx$. (See Art. 204.) Also, $\bar{y}=\dfrac{\frac{1}{2}\int y^2\,dx}{A}$. (See Art. 323.)

$$\therefore\ V=A\cdot 2\pi\bar{y}.$$

We have thus far supposed the bodies considered to be homogeneous. When they are non-homogeneous, the formulas for C.G. differ from those given above only by the presence of a factor ρ in each integrand.

EXERCISES

1. Express, by single integrals, the coördinates of the C.G. of a homogeneous plane area when the equation of the bounding curve is given in rectangular coördinates.

2. Find the coördinates of the C.G. of a segment of the parabola $y^2 = 4\,ax$ cut off by the chord $x = h$.

Ans. $\bar{x} = \frac{3}{5}h,\ \bar{y} = 0.$

3. Find the coördinates of the C.G. of the area of a semicircle of radius a.

Ans. $\bar{x} = \dfrac{4\,a}{3\,\pi},\ \bar{y} = 0.$

4. Find the coördinates of the C.G. of a semi-ellipse, semi-axes a and b, the bisecting line being the major axis.

Ans. $\bar{x} = \dfrac{4\,b}{3\,\pi},\ y = 0.$

5. Find the coördinates of the C.G. of the plane area bounded by the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, which lies in the first quadrant.

Ans. $\bar{x} = \bar{y} = \dfrac{256\,a}{315\,\pi}.$

6. Find the coördinates of the C.G. of the cycloid

$$\bar{x} = a(\theta - \sin\theta),\ \bar{y} = a(1 - \cos\theta).$$

Ans. $\bar{x} = a\pi,\ \bar{y} = \frac{5}{6}\,a.$

7. Find the coördinates of the C.G. of one loop of the lemniscate $r^2 = a^2 \cos 2\,\theta$.

Ans. $\bar{x} = \dfrac{\pi\sqrt{2}}{8}\,a,\ \bar{y} = 0.$

8. Find the coördinates of the C.G. of a circular sector of angle $2\,\alpha$.

Ans. If the radius bisecting the sector is on the x-axis,

$$\bar{x} = \frac{2}{3}\,\frac{a \sin\alpha}{\alpha},\ \bar{y} = 0.$$

9. Find the coördinates of the C.G. of the cardioide $r = a(1 + \cos\theta)$ on the right side of the line through the pole perpendicular to the initial line.

Ans. $\bar{x} = \dfrac{16 + 5\,\pi}{16 + 6\,\pi}\,a,\ \bar{y} = \dfrac{10\,a}{8 + 3\,\pi}.$

10. Find the coördinates of the C.G. of the arc of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ in the first quadrant. *Ans.* $\bar{x} = \bar{y} = \frac{2}{5}a.$

11. Find the coördinates of the C.G. of an arc of a semi-cycloid, the equations of the cycloid being $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. *Ans.* $\bar{x} = (\pi - \frac{4}{3})a,\ \bar{y} = -\frac{2}{3}a.$

12. Find the coördinates of the C.G. of the solid formed by revolving the sector of a circle, angle 2α, about one of its extreme radii.

Ans. If an extreme radius is on the x-axis, $\bar{x} = \frac{3}{4}a\cos^2\alpha,\ \bar{y} = 0.$

13. Find the coördinates of the C.G. of the segment of the paraboloid, formed by revolving the parabola $y^2 = 4ax$ about the x-axis, cut off by the plane $x = h$. *Ans.* $\bar{x} = \frac{2}{3}h,\ \bar{y} = 0.$

14. Find the coördinates of the C.G. of the surface formed by revolving the cardioide $r = a(1 + \cos\theta)$ about the initial line.

Ans. $\bar{x} = \frac{50}{63}a,\ \bar{y} = 0.$

15. Find the coördinates of the C.G. of a hemispherical surface. *Ans.* $\left(\frac{a}{2}, 0\cdot\right)$

16. Find the coördinates of the C.G. of a hemisphere whose density varies as the distance from the center of the sphere.

Ans. $(\frac{2}{5}a, 0.)$

17. Find the distance between the center and the C.G. of one half an anchor ring generated by a circle of radius a whose center describes a circle of radius b. *Ans.* $\frac{4b^2 + a^2}{2\pi b}.$

18. Find the coördinates of the C.G. of a volume when the equation of the bounding surface is given in rectangular coördinates.

$$Ans.\quad \bar{x} = \frac{\iiint x\,dx\,dy\,dz}{\text{vol.}},\ \bar{y} = \frac{\iiint y\,dx\,dy\,dz}{\text{vol.}},$$

$$\bar{z} = \frac{\iiint z\,dx\,dy\,dz}{\text{vol.}}.$$

19. A cone of height h is cut out of a cylinder of the same base and height. Find the distance of the C.G. of the remainder from the vertex. *Ans.* $\frac{3}{8}h$.

20. Find the coördinates of the C.G. of the surface of a solid of revolution when the equation of the bounding curve is given in polar coördinates.

$$Ans.\quad \bar{x} = \frac{\int r^2 \cos\theta \sin\theta \, ds}{\int r \sin\theta \, ds}, \quad \bar{y} = 0.$$

21. Prove that the $\bar{x}$ of the C.G. of a set of n plane areas in the same plane is

$$\bar{x} = \frac{\bar{x}_1 F_1 + \bar{x}_2 F_2 + \bar{x}_3 F_3 + \cdots + \bar{x}_n F_n}{F_1 + F_2 + F_3 + \cdots + F_n},$$

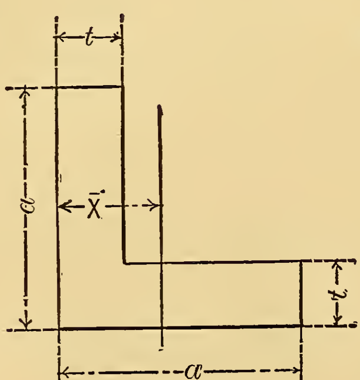
where $\bar{x}_k$ is the $\bar{x}$ of the C.G. of the area F_k.

22. Find the coördinates of the C.G. of the area between a quadrant of a circle and the circumscribing square.

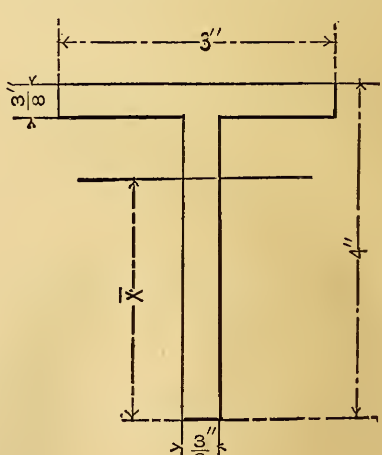
$$Ans.\quad \bar{x} = \bar{y} = \frac{2a}{3(4-\pi)}.$$

Find the position of the C.G. of each of the following areas:

23.



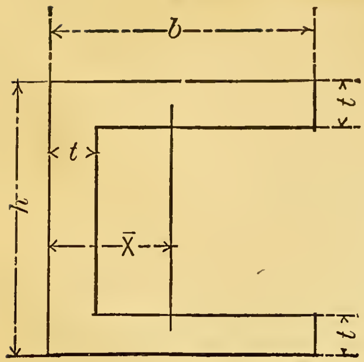
24.



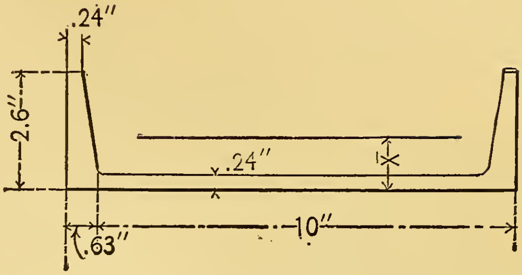
23. *Ans.* $\bar{x} = \dfrac{a^2 + at - t^2}{2(2a - t)}$.

24. *Ans.* $\bar{x} = 2.718''$.

25.



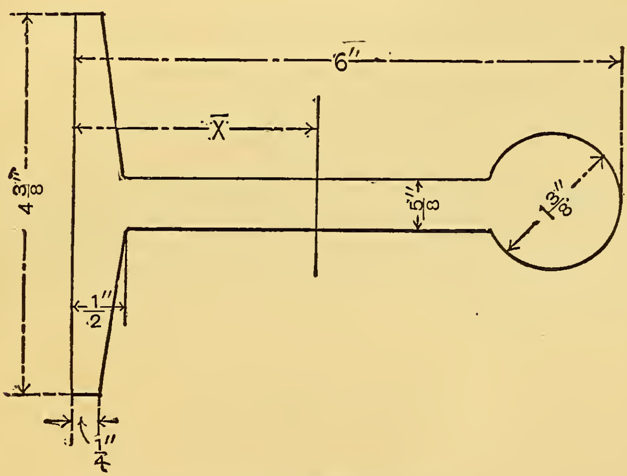
26.



25. *Ans.* $\bar{x} = \dfrac{ht - 2t^2 + 2b^2}{2h - 4t + 4b}$.

26. *Ans.* $\bar{x} = 0.638''$.

27.



27. *Ans.* $\bar{x} = 2.568''$.

Find, by Pappus' Theorem:

28. The surface and volume of a sphere.

29. The surface and volume of the torus generated by the curve $x^2 + (y - b)^2 = a^2$, $b > a$.

30. The surface and volume of the solid formed by revolving the cycloid about its base.

CHAPTER XXXVII

MOMENTS OF INERTIA

327. **Definition.** The **moment of inertia** of a particle with reference to a point, line, or plane is the product of the mass of the particle and the square of the distance of the particle from the point, line, or plane.

Thus, if m be the mass of a particle and r the distance of the particle from a point, line, or plane, the moment of inertia of the particle with reference to the point, line, or plane is mr^2.

328. Suppose that there are n particles of masses m_1, m_2, m_3, $\cdots$, m_n situated at points A_1, A_2, A_3, $\cdots$, A_n respectively. To determine the moment of inertia of the system with reference to the coördinate axes.

Let the coördinates of A_1, A_2, A_3, $\cdots$, A_n be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), $\cdots$, (x_n, y_n, z_n) respectively. Let the moments of inertia of the system with reference to the x, y, and z axes be denoted by I_x, I_y, I_z respectively.

Then

$$I_x = m_1(y_1^2 + z_1^2) + m_2(y_2^2 + z_2^2) + m_3(y_3^2 + z_3^2) + \cdots + m_n(y_n^2 + z_n^2)$$
$$= \sum m(y^2 + z^2).$$

$$I_y = m_1(x_1^2 + z_1^2) + m_2(x_2^2 + z_2^2) + m_3(x_3^2 + z_3^2) + \cdots + m_n(x_n^2 + z_n^2)$$
$$= \sum m(x^2 + z^2).$$

$$I_z = m_1(x_1^2 + y_1^2) + m_2(x_2^2 + y_2^2) + m_3(x_3^2 + y_3^2) + \cdots + m_n(x_n^2 + y_n^2)$$
$$= \sum m(x^2 + y^2).$$

329. The moment of inertia of a continuous body with reference to a point, line, or plane can be found approximately by dividing the body up into infinitesimal elements of mass, Δm, forming the product of each element of mass and the square of the distance of some point in it from the given point, line, or plane, and taking the sum of the results.

Definition. The limit which this sum approaches as Δm approaches zero is the moment of inertia of the body with reference to the point, line, or plane.

Thus, the moments of inertia of a continuous body with reference to the coördinate axes are, approximately:

$$I_x = \sum (y^2 + z^2)\Delta m,$$

$$I_y = \sum (x^2 + z^2)\Delta m,$$

$$I_z = \sum (x^2 + y^2)\Delta m,$$

where x, y, and z are the distances of some point in the elements of mass, Δm, from the x, y, and z-axis respectively, and the summation includes all the elements of mass Δm. The moments of inertia of the body with reference to the axes are:

$$I_x = \int (y^2 + z^2)dm,$$

$$I_y = \int (x^2 + z^2)dm,$$

$$I_z = \int (x^2 + y^2)dm,$$

where the limits embrace the whole solid.

330. As illustrations of the method of determining the moment of inertia of a continuous body, consider the following examples:

Example 1. Find the moment of inertia of a homogeneous rod of length l and density ρ with reference to an axis perpendicular to the rod through one end.

Let AB (Fig. 135) be the given rod, and A the end through which the axis passes.

FIG. 135.

Divide AB into n equal parts. Call each part Δx.

Let A_kA_{k+1} be a length Δx where A_k is distant x_k from A.

Therefore the required moment of inertia $= \underset{n=\infty}{\text{limit}} \sum_{x=0}^{x=l} x^2\rho\Delta x.$

$$= \rho\int_0^l x^2dx.$$

$$= \rho\frac{l^3}{3}.$$

If M is the mass of the rod, the required moment of inertia $= \frac{M}{3}l^2.$

EXAMPLE 2. Find the moment of inertia of a homogeneous right circular cone of height h and radius of base a with reference to its axis.

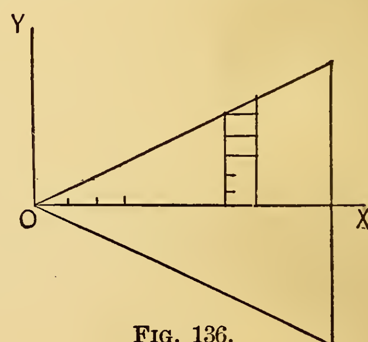


FIG. 136.

Divide the volume of the cone up into infinitesimal elements in the same manner as if a volume were sought by double integration. (See Art. 214.)

The element of volume

$$= 2\pi[y_l\Delta y\Delta x + \tfrac{1}{2}\Delta y^2\Delta x].$$

Therefore the required moment of inertia

$$= \underset{n=\infty}{\text{limit}} \sum_{x=0}^{x=h} \left\{ \left[2\pi\rho \underset{m=\infty}{\text{limit}} \sum_{y=0}^{y=\frac{a}{h}x} y^3\Delta y \right] \Delta x \right\}$$

$$= 2\pi\rho\int_0^h\int_0^{\frac{a}{h}x} y^3\,dx\,dy,$$

$$= \frac{\pi\rho}{10}a^4h = \frac{3}{10}Ma^2, \text{ where } M \text{ is the mass of the cone.}$$

331. Theorem. The moment of inertia of a body with reference to any line is equal to the moment of inertia of the body with reference to a parallel line through the center of gravity, *plus* the product of the mass of the body and the square of the distance between the two lines.

Proof. Draw a plane through the given line and the center of gravity of the body. Take the origin at the center of gravity, the x-axis parallel to the given line, and the y-axis in the plane containing the x-axis and the given line.

Let I_x denote the moment of inertia of the body with reference to the x-axis, I_l the moment of inertia of the body with reference to the given line, M the mass of the body, and a the distance between the given line and the x-axis.

Then $$I_x = \int (y^2 + z^2)\, dm, \text{ and } I_l = \int \{(a \pm y)^2 + z^2\}\, dm.$$

$$\therefore\ I_l - I_x = a^2 \int dm \pm 2a \int y\, dm,$$

$$= a^2 M \pm 2a \int y\, dm.$$

Now $\int y\, dm = 0$, since the origin is at the center of gravity of the body. (See Art. 319.)

$$\therefore\ I_l - I_x = a^2 M.$$

$$\therefore\ I_l = I_x + a^2 M,$$

which establishes the theorem.

332. A quantity r_0 can always be found such that $r_0{}^2 = \dfrac{\int r^2 dm}{M}$, where M is the mass of the body, and $\int r^2 dm$ the moment of inertia of the body with reference to some point, line, or plane.

Definition. This quantity r_0 such that $r_0{}^2 = \dfrac{\int r^2 dm}{M}$ is called the **radius of gyration** of the body with reference to the point, line, or plane.

The radius of gyration determines a point distant from a point, line, or plane such that if the whole mass of the body be supposed concentrated at that point, the moment of inertia of the body would be unchanged.

EXERCISES

Find the moment of inertia:

1. Of a parallelogram of base b and altitude h, with reference to the base; with reference to a line through the C.G. parallel to the base.

Ans. $I_b = \dfrac{bh^3}{3}$; $I_g = \dfrac{bh^3}{12}$.

2. Of a triangle with reference to the base; with reference to a line through the C.G. parallel to the base; with reference to a line through the vertex parallel to the base.

Ans. $I_b = \dfrac{bh^3}{12}$; $I_g = \dfrac{bh^3}{36}$; $I_v = \dfrac{bh^3}{4}$.

3. (*a*) Of a square with reference to a diagonal.

(*b*) Of a circle with reference to a coördinate axis; with reference to the origin.

Ans. (*a*) $I = \dfrac{a^4}{12}$. (*b*) $I_a = \dfrac{\pi a^4}{4}$; $I_0 = \dfrac{\pi a^2}{2}$.

4. Of the area in Exercise 23, Chapter XXXVI, with reference to the base; with reference to a line through the C.G. parallel to the base.

Ans. $I_b = \dfrac{t}{3}(a^3 + at^2 - t^3)$; $I_g = \dfrac{t}{12}\,\dfrac{5\,a^4 - 10\,a^3t + 11\,a^2t^2 - 6\,at^3 + t^4}{2\,a - t}$.

5. Prove that the moment of inertia of a system of areas or solids with reference to any axis is equal to the sum of the moments of inertia of the separate areas or solids with reference to that axis.

Find the moments of inertia with reference to the horizontal and vertical gravity axes of the areas described in:

6. Exercise 24, Chapter XXXVI.

Ans. $I_x = 3.963$ bi. in; $I_y = 0.8597$ bi. in.

7. Exercise 25, Chapter XXXVI.

$$Ans.\ I_x = \tfrac{1}{12}bh^3 - \tfrac{1}{12}(b-t)(h-2t)^3;$$

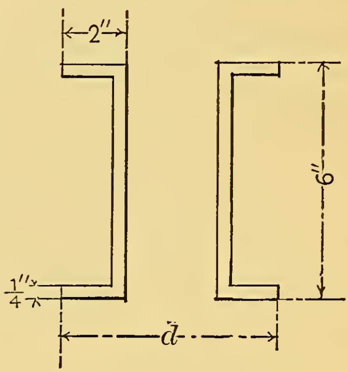
$$I_y = \tfrac{1}{3}(h-2t)t^3 + \tfrac{2}{3}tb^3 - \frac{t(ht+2b^2-2t^2)^2}{4(h+2b-2t)}.$$

8. Exercise 26, Chapter XXXVI.

Ans. $I_x = 2.298$ bi. in.; $I_y = 66.72$ bi. in.

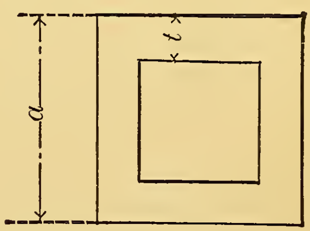
9. Exercise 27, Chapter XXXVI.

Ans. $I_x = 2.51$ bi. in.; $I_y = 24.65$ bi. in.



10. Determine the distance d so that the moment of inertia of the two areas of the figure shall be the same about the horizontal and vertical gravity axes.

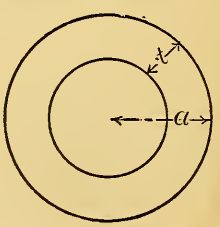
Ans. $d = 7.309''$.



11. Find the moment of inertia and radius of gyration of the hollow square of the figure with reference to a line through the C.G. and parallel to a side.

$$Ans.\ I = \tfrac{1}{12}\{a^4 - (a-2t)^4\};$$

$$k^2 = \tfrac{1}{12}\{a^2 + (a-2t)^2\}.$$



12. Find the moment of inertia and radius of gyration of the area of the ring of the figure with reference to the center; with reference to a diameter.

$$Ans.\ I_c = \frac{\pi}{2}\{r^4 - (r-t)^4\};\ k^2 = \tfrac{1}{2}\{r^2 + (r-t)^2\}.$$

$$I_a = \frac{\pi}{4}\{r^4 - (r-t)^4\};\ k^2 = \tfrac{1}{4}\{r^2 + (r-t)^2\}.$$

13. Find the radius of gyration of a homogeneous circular wire with reference to its diameter.

Ans. $k = \dfrac{r}{\sqrt{2}}$.

14. Find the moment of inertia of the homogeneous ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, with reference to the x-axis.

Ans. $I = \dfrac{M}{5}(b^2 + c^2)$.

15. Find the moment of inertia of a homogeneous sphere of radius a with reference to a diameter.

Ans. $I = \frac{2}{5} Va^2 m$, where m is the mass per unit of volume.

16. Find the moment of inertia and radius of gyration of a homogeneous right circular cylinder of length l and radius r, with reference to its axis; with reference to a diameter of one end.

Ans. $I_{\text{axis}} = \frac{1}{2} MR^2$; $k = \dfrac{R}{\sqrt{2}} \cdot I_{\text{diam.}} = M\left(\dfrac{l^2}{3} + \dfrac{R^2}{4}\right)$.

17. Show that if either coördinate axis is an axis of symmetry of an area F, then $\int\int xy\, dx\, dy$ is zero, the limits being taken to embrace the whole area.

18. If I_α is the moment of inertia of an area with reference to a line through the origin making an angle α with the x-axis, and if either coördinate axis is an axis of symmetry, then

$$I_\alpha = \cos^2 \alpha\, I_x + \sin^2 \alpha\, I_y.$$

19. In Exercise 18, show that if $I_x = I_y$, and either axis is an axis of symmetry, then I_α is the same for all values of α.

20. Prove that the least radius of gyration of the figure in Exercise 23, Chapter XXXVI, with reference to any gravity axis is for the axis sloped 45° to the sides.

INDEX

[The numbers refer to the pages.]

Date Due

Mar. 17			
MAR 11 '57			
JAN 8 '59 M			
JAN 3 '62			
[illegible] 27 '62			
[illegible] 16 '62			
MA[illegible] '62			
[illegible]			
JE 12 '64			
FE 27 '69			
OCT 1 0 1997			